Vector Algebra

- The quantity which involves only one value, i.e. magnitude, is called a scalar quantity. For example: time, mass, distance, energy, etc.
- The quantity which has both magnitude and a direction is called a vector quantity. For example: force, momentum, acceleration, etc.
- A line with a direction is called a directed line. Let $\overrightarrow{AB}$ be a directed line along direction B.

Here,

  - The length of the line segment AB represents the magnitude of the above directed line. It is denoted by $\left|\overrightarrow{AB}\right|$ or $|\vec{a}|$ or *a*.
  - $\overrightarrow{AB}$ represents the vector in the direction towards point B. Therefore, the vector represented in the above figure is $\overrightarrow{AB}$. It can also be denoted by $\vec{a}$.
  - The point A from where the vector $\overrightarrow{AB}$ starts is called its initial point and the point B where the vector $\overrightarrow{AB}$ ends is called its terminal point.

- The angles *a*, *b*, and *g* made by the vector $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ with the positive directions of the *x*-axis, *y*-axis, and *z*-axis respectively are called its direction angles. The cosines of the angle made by the vector $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ with the positive directions of *x*, *y*, and *z* axes are its direction cosines. These are usually denoted by *l* = cos*a*, *m* = cos*b*, and *n* = cos*g*. Also, $l^2 + m^2 + n^2 = 1$

Example: Write the direction ratio's of the vector $\vec{r} = 2\hat{i} - \hat{j} - 2\hat{k}$ and hence calculate its direction cosines.

Solution: The direction ratio's *a, b, c* of a vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ are the respective components *x, y* and *z* of the vector.
The direction ratio's of the given vector are $a = 2$, $b = -1$ and $c = -2$
If *l, m* and *n* are the direction cosines of the given vector, then

$$l = \frac{a}{|\vec{r}|}, m = \frac{b}{|\vec{r}|}, n = \frac{c}{|\vec{r}|}$$

$$|\vec{r}| = \sqrt{(2)^2 + (-1)^2 + (-2)^2} = \sqrt{9} = 3$$

$$\therefore l = \frac{2}{3}, m = \frac{-1}{3} \text{ and } n = \frac{-2}{3}$$

- The direction cosines (*l*, *m*, *n*) of a vector $a\hat{i} + b\hat{j} + c\hat{k}$ are

$l = \frac{a}{r}, m = \frac{b}{r}, n = \frac{c}{r}$, where r = magnitude of the vector $a\hat{i} + b\hat{j} + c\hat{k}$

- The various types of vectors are given as follows:
    - Zero vector: A vector whose initial and terminal points coincide is called a zero vector (or null vector). It is denoted as $\vec{0}$. The vectors $\overrightarrow{AA}$, $\overrightarrow{BB}$ represent zero vectors.
    - Unit vector: A vector whose magnitude is unity, i.e. 1 unit, is called a unit vector. The unit vector in the direction of any given vector $\vec{a}$ is denoted by $\hat{a}$ and it is calculated by

Note: that if *l*, *m*, and *n* are direction cosines of a vector, then $l\hat{i} + m\hat{j} + n\hat{k}$ is the unit vector in the direction of that vector.

Example: To find the unit vector along the direction of a vector $\vec{r} = 16\hat{i} - 15\hat{j} + 12\hat{k}$, we may proceed as follows:

- The position vector of a point P(x, y, z) with respect to the origin (0, 0, 0) is given by $\overrightarrow{OP} = x\hat{i} + y\hat{j} + z\hat{k}$. This form of any vector is known as the component form.

  Here,

  - $\hat{i}, \hat{j}$, and $\hat{k}$ are called the unit vectors along the x-axis, y-axis, and z-axis respectively.
  - x, y, and z are the scalar components (or rectangular components) along x-axis, y-axis, and z-axis respectively.
  - $x\hat{i} + y\hat{j} + z\hat{k}$ are called vector components of $\overrightarrow{OP}$ along the respective axes.
  - The magnitude of $\overrightarrow{OP}$ is given by $\left|\overrightarrow{OP}\right| = \sqrt{x^2 + y^2 + z^2}$
- The scalar components of a vector are its direction ratios and represent its projections along the respective axes.

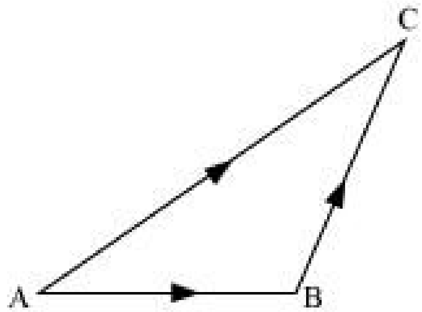
  The direction ratios of a vector $\vec{p} = a\hat{i} + b\hat{j} + c\hat{k}$ are a, b, and c.

  Here, a, b, and c respectively represent projections of $\vec{p}$ along x-axis, y-axis, and z-axis.

- The sum of two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is given by,

  $\vec{a} + \vec{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$

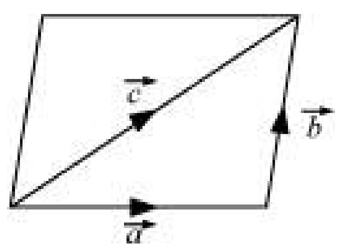
- The difference of two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ is given by
  $\vec{a} - \vec{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$
- **Triangle law of vector addition:** If two vectors are represented by two sides of a triangle in order, then the third closing side of the triangle in the opposite direction of the order represents the sum of the two vectors.



$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

Note: The vector sum of the three sides of a triangle taken in order is $\vec{0}$.

- **Parallelogram law of vector addition:** If two vectors are represented by two adjacent sides of a parallelogram in order, then the diagonal of the parallelogram in the opposite direction of the order represents the sum of two vectors.



$\vec{c} = \vec{a} + \vec{b}$

- The properties of vector addition are given as follows:
  - Commutative property: $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

- Associative property: $\vec{a} + (b + c) = (a + b) + \vec{c}$
- Existence of additive identity: The vector $\vec{0}$ is additive identity of a vector $\vec{a}$, since $\vec{a} + \vec{0} = \vec{0} + \vec{a} = \vec{a}$
- Existence of additive inverse: The vector $-\vec{a}$ is called additive inverse of $\vec{a}$, since $\vec{a} + (-\vec{a}) = (-\vec{a}) + \vec{a} = 0$

- The multiplication of vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ by any scalar l is given by,

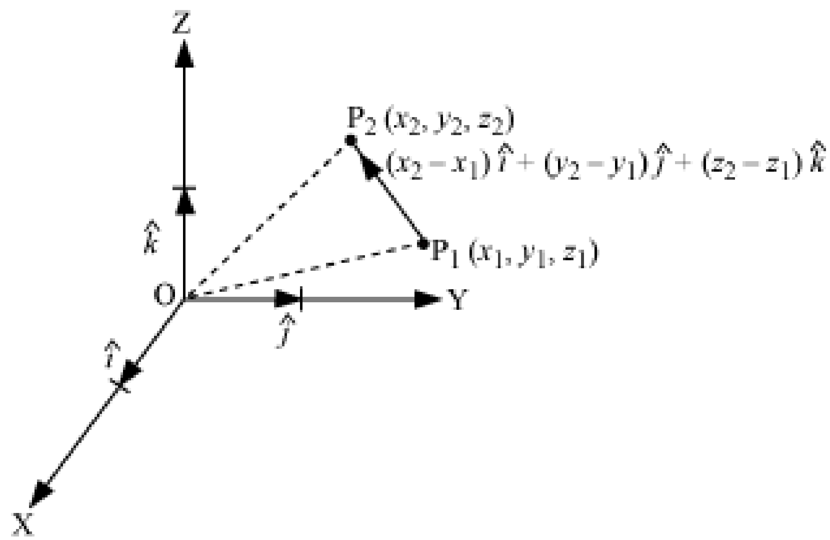
  $\lambda\vec{a} = (\lambda a_1)\hat{i} + (\lambda a_2)\hat{j} + (\lambda a_3)\hat{k}$

- The magnitude of the vector $\lambda\vec{a}$ is given by $|\lambda\vec{a}| = |\lambda||\vec{a}|$
- The vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are equal, if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$
- Let $\vec{a_1}$ and $\vec{a_2}$ be two vectors, and k_1 and k_2 be any scalars, then the following are the distributive laws of addition and multiplication of a vector by a scalar:
  - $k_1\vec{a_1} + k_2\vec{a_1} = (k_1 + k_2)\vec{a_1}$
  - $k_1(k_2\vec{a_1}) = (k_1k_2)\vec{a_1}$
  - $\vec{k_1}(\vec{a_1} + \vec{a_2}) = k_1a_1 + k_1a_2$
- Collinear vectors:
  - Two vectors $\vec{a}$ and $\vec{b}$ are collinear, if and only if there exists a non-zero scalar l such that
  - Two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear, if and only if

Vector Joining Two Points

The vector joining two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, represented as $\overrightarrow{P_1P_2}$, is calculated as

$$\overrightarrow{P_1P_2} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$$



The magnitude of $\overrightarrow{P_1P_2}$ is given by $\left|\overrightarrow{P_1P_2}\right| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Section Formula

If point R (position vector $\vec{r}$) lies on the vector $\overrightarrow{PQ}$ joining two points P (position vector $\vec{a}$) and Q (position vector $\vec{b}$) such that R divides $\overrightarrow{PQ}$ in the ratio m: n $\left[\text{i.e.} \frac{\overrightarrow{PR}}{\overrightarrow{RQ}} = \frac{m}{n}\right]$

Internally, then $\vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

Externally, then $\vec{r} = \dfrac{m\vec{b} - n\vec{a}}{m-n}$

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- The scalar product of two non-zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \cdot \vec{b}$ and it is given by the formula $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$, where *q* is the angle between $\vec{a}$ and $\vec{b}$ such that 0 ≤ *q* ≤ p

If either $\vec{a} = 0$ or $\vec{b} = 0$, then in this case, *θ* is not defined and $\vec{a} \cdot \vec{b} = 0$

- The following are the observations related to the scalar product of two vectors:
  - $\vec{a} \cdot \vec{b}$ is a real number.
  - The angle *q* between vectors $\vec{a}$ and $\vec{b}$ is given by,

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right)$$

  - Let $\vec{a}$ and $\vec{b}$ be any two non-zero vectors , then $\vec{a} \cdot \vec{b} = 0$, if and only if $\vec{a} \perp \vec{b}$

- If $q = 0$, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$
- If $q = $ p, then $\vec{a} \cdot \vec{b} = -|\vec{a}||\vec{b}|$
- $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1, \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- The properties of scalar product are as follows:
  - Commutative property: $\hat{a} \cdot \hat{b} = \hat{b} \cdot \hat{a}$
  - Distributivity of scalar product over addition: $\hat{a} \cdot (\hat{b} + \hat{c}) = \hat{a} \cdot \hat{b} + \hat{a} \cdot \hat{c}$

Example: Find the angle between the vectors $8\hat{i} - 4\hat{j} - \hat{k}$ and $3\hat{i} - 6\hat{j} + 2\hat{k}$.

Solution:

Let $\vec{a} = 8\hat{i} - 4\hat{j} - \hat{k}$

$\vec{b} = 3\hat{i} - 6\hat{j} + 2\hat{k}$

Angle between $\vec{a}$ and $\vec{b}$ is given by,

$$\theta = \cos^{-1}\left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}\right)$$

However, $\vec{a} \cdot \vec{b} = 8\times3 + (-4)\times(-6) + (-1)\times2 = 46$

$$|\vec{a}| = \sqrt{(8)^2 + (-4)^2 + (-1)^2} = 9$$

$$|\vec{b}| = \sqrt{(3)^2 + (-6)^2 + (2)^2} = 7$$

$$\therefore \theta = \cos^{-1}\left(\frac{46}{9\times7}\right) = \cos^{-1}\left(\frac{46}{63}\right)$$

- Projection of a vector:
    - If $\hat{p}$ is the unit vector along a line *l*, then the projection of a vector $\vec{a}$ on the line *l* is given by $\vec{a} \cdot \hat{p}$.
    - Projection of a vector $\vec{a}$ on other vector $\vec{b}$ is given by $\vec{a} \cdot \hat{b}$ or $\frac{\vec{a} \cdot \vec{b}}{\left|\vec{b}\right|}$.

Example: Find the projection of the vector $3\hat{i} - 8\hat{j} + 6\hat{k}$ on the vector $2\hat{i} - 3\hat{j} - 6\hat{k}$.

Solution:

Let $\vec{a} = 3\hat{i} - 8\hat{j} + 6\hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} - 6\hat{k}$

Then, the projection of $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{\vec{a} \cdot \vec{b}}{\left|\vec{b}\right|} = \frac{\left(3\hat{i} - 8\hat{j} + 6\hat{k}\right) \cdot \left(2\hat{i} - 3\hat{j} - 6\hat{k}\right)}{\sqrt{(2)^2 + (-3)^2 + (-6)^2}}$$

$$= \frac{6 + 24 - 36}{7}$$

$$= -\frac{6}{7}$$

- The vector product (or cross product) of two non-zero vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ and is defined by $\vec{a} \times \vec{b} = \left|\vec{a}\right| \left|\vec{b}\right| \sin\theta \hat{n}$, where θ is the angle between $\vec{a}$ and $\vec{b}$, $0 \leq \theta \leq \pi$, and $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$.

- If $\vec{a} = a_1\hat{i} - a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} - b_2\hat{j} + b_3\hat{k}$ are two vectors, then their cross product $\vec{a} \times \vec{b}$, is defined by $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
- The following are the observations made by the vector product of two vectors:
  - $\vec{a} \times \vec{b} = \vec{0}$, if and only if $\vec{a} \parallel \vec{b}$
  - $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

    $\hat{i}\times\hat{j} = \hat{k}, \hat{j}\times\hat{k} = \hat{i}, \hat{k}\times\hat{i} = \hat{j}$

    $\hat{j}\times\hat{i} = -\hat{k}, \hat{k}\times\hat{j} = -\hat{i}, \hat{i}\times\hat{k} = -\hat{j}$
  - In terms of vector product, the angle θ between two vectors $\vec{a}$ and $\vec{b}$ is given by $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$ or
  - If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a triangle, then its area is given as $\frac{1}{2}|\vec{a} \times \vec{b}|$.

Example:
Find the area of a triangle having the points A (1, 2, 3), B (1, –1, – 3) and C (–1, 1, 2) as its vertices

Solution:

$$\overrightarrow{AB} = (1-1)\hat{i} + (-1-2)\hat{j} + (-3-3)\hat{k} = -3\hat{j} - 6\hat{k}$$

$$\overrightarrow{AC} = (-1-1)\hat{i} + (1-2)\hat{j} + (2-3)\hat{k} = -2\hat{i} - \hat{j} - \hat{k}$$

The area of the given triangle is $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -3 & -6 \\ -2 & -1 & -1 \end{vmatrix}$$

$$= \hat{i}\left(3-6\right) - \hat{j}\left(0-12\right) + \hat{k}(0-6)$$

$$= -3\hat{i} + 12\hat{j} - 6\hat{k}$$

$$\therefore \left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = \sqrt{\left(-3\right)^2 + \left(12\right)^2 + (-6)^2} = \sqrt{9+144+36} = \sqrt{189}$$

Thus, the required area is $\frac{1}{2}\sqrt{189}$.

- If $\vec{a}$ and $\vec{b}$ represent the adjacent sides of a parallelogram, then its area is given as $\left|\vec{a}\times\vec{b}\right|$.

- The properties of vector product are as follows:
  - Not commutative: $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$

    However, $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$

  - Distributivity of vector product over addition:

- $\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
- $\lambda\left(\vec{a} \times \vec{b}\right) = (\lambda \vec{a}) \times \vec{b} = \vec{a} \times \left(\lambda \vec{b}\right)$

Example: If the position vectors of vertices P, Q, R, and S of quadrilateral PQRS are $-\hat{i} + 2\hat{j} + \hat{k}$, $\hat{i} - 2\hat{j} + 5\hat{k}$, $4\hat{i} - 7\hat{j} + 8\hat{k}$, and $2\hat{i} - 3\hat{j} + 4\hat{k}$ respectively, then find the area of quadrilateral PQRS.

Solution:

$$\overrightarrow{PQ} = (1+1)\hat{i} + (-2-2)\hat{j} + (5-1)\hat{k} = 2\hat{i} - 4\hat{j} + 4\hat{k}$$

$$\overrightarrow{QR} = (4-1)\hat{i} + (-7+2)\hat{j} + (8-5)\hat{k} = 3\hat{i} - 5\hat{j} + 3\hat{k}$$

$$\overrightarrow{RS} = (2-4)\hat{i} + (-3+7)\hat{j} + (4-8)\hat{k} = -2\hat{i} + 4\hat{j} + 4\hat{k}$$

$$= -\left(2\hat{i} - 4\hat{j} + 4\hat{k}\right)$$

$$= -\overrightarrow{RS}$$

$$\overrightarrow{SP} = (-1-2)\hat{i} + (2+3)\hat{j} + (1-4)\hat{k} = -3\hat{i} + 5\hat{j} - 3\hat{k}$$

$$= -\left(3\hat{i} - 5\hat{j} + 3\hat{k}\right)$$

$$= -\overrightarrow{QR}$$

Clearly, $\overrightarrow{PQ} \parallel \overrightarrow{RS}$ and $\overrightarrow{QR} \parallel \overrightarrow{SP}$. Hence, PQRS is a parallelogram.

Therefore, area (PQRS) = $\left|\overrightarrow{PQ} \times \overrightarrow{QR}\right|$

Now,

$$\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 4 \\ 3 & -5 & 3 \end{vmatrix}$$

$$= (-12 + 20)\hat{i} - (6 - 12)\hat{j} + (-10 + 12)\hat{k}$$

$$= 8\hat{i} + 6\hat{j} + 2\hat{k}$$

$$\therefore \left|\overrightarrow{PQ} \times \overrightarrow{QR}\right| = \sqrt{(8)^2 + (6)^2 + (2)^2} = 2\sqrt{26}$$

Hence, area of the quadrilateral PQRS is $2\sqrt{26}$ square units.