

Differential Equations

- An equation is called a differential equation, if it involves variables as well as derivatives of dependent variable with respect to independent variable.

For example:

$$x \frac{d^4 y}{dx^4} + y \left(\frac{d^2 y}{dx^2} \right)^3 - 2x^2 y \frac{dy}{dx} + 3 = 0$$

is a differential equation.

Sometimes, we may write $\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \frac{d^3 y}{dx^3}, \frac{d^4 y}{dx^4}$ etc. as y', y'', y''', y'''' etc. respectively. Also, note that we cannot say that $\tan(y') + x = 0$ is a differential equation.

- Order of a differential equation is defined as the order of the highest order derivative of dependent variable with respect to independent variable involved in the given differential equation.

For example: The highest order derivative present in the differential equation

$$x^3 y^5 y'''' - 3x^2 y'' + xy y' - 5 = 0$$

is y'''' . Therefore, the order of this differential equation is 4.

- Degree of a differential equation is the highest power of the highest order derivative in it.

For example: The degree of the differential equation $(y'')^2 - 2x(y'')^5 - xy(y'')^2 + y' = 0$ is 2, since the highest power of the highest order derivative, y'' , is 2.

Note: The degree of the differential equation $\frac{d^2 y}{dx^2} + \cos\left(\frac{dy}{dx}\right) = 0$ is not defined since it is not a polynomial equation in $\frac{dy}{dx}$. However, its order is 2.

- If a differential equation is defined, then its order and degree are always positive integers.

- A function that satisfies the given differential equation is called a solution of a given differential equation.

Example: Verify whether $y = \sin x + \cos x - 5$ is a solution of the differential equation $y''' + y' = 0$ or not.

Solution:

We have, $y = \sin x + \cos x - 5$

$$\therefore y' = \cos x - \sin x$$

$$y' = -\sin x - \cos x = -(\sin x + \cos x)$$

$$y'' = -(\cos x - \sin x) = -y'$$

$$\Rightarrow y''' + y' = 0$$

Therefore, $y = \sin x + \cos x - 5$ is a solution of the differential equation $y''' + y' = 0$

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- To form a differential equation from a given function, we differentiate the function successively as many times as the number of arbitrary constants in the given function and then eliminate the arbitrary constants.

Example: Form the differential equation, representing the family of circle

$$(x - a)^2 + (y - b)^2 = r^2, \text{ where } a \text{ and } b \text{ are arbitrary constants.}$$

Solution:

$$\text{We have } (x - a)^2 + (y - b)^2 = r^2 \quad (1)$$

Differentiating with respect to x , we obtain

$$2(x - a) + 2(y - b)\frac{dy}{dx} = 0 \quad \dots(2)$$

Again differentiating with respect to x , we obtain

$$2 + 2 \left[\left(\frac{dy}{dx} \right)^2 + (y - b) \frac{d^2 y}{dx^2} \right] = 0$$

$$\Rightarrow (y - b) \frac{d^2 y}{dx^2} = \left[1 + \left(\frac{dy}{dx} \right)^2 \right]$$

$$\Rightarrow y - b = - \left[\frac{1 + (y')^2}{y''} \right]$$

Substituting $y - b = - \frac{[1 + (y')^2]}{y''}$ in equation (2), we obtain

$$2(x - a) - 2 \frac{[1 + (y')^2]}{y''} \times y' = 0$$

$$\Rightarrow x - a = \frac{y' + (y')^3}{y''}$$

Substituting the values of $(x - a)$ and $(y - b)$ in equation (1), we obtain

$$\begin{aligned}
& \left[\frac{y' + (y')^3}{y''} \right]^2 + \left[-\frac{1 + (y')^2}{y''} \right]^2 = r^2 \\
& \Rightarrow (y')^2 \left[1 + (y')^2 \right]^2 + \left[1 + (y')^2 \right]^2 = r^2 (y'')^2 \\
& \Rightarrow \left[1 + (y')^2 \right]^2 \left[1 + (y')^2 \right] = r^2 (y'')^2 \\
& \Rightarrow \left[1 + (y')^2 \right]^3 - r^2 (y'')^2 = 0
\end{aligned}$$

This is the required differential equation of the given circle.

- The three methods of solving first order, first degree differential equations are given as follows:
 - **Variable separable method:** This method is used to solve such an equation in which variables can be separated completely i.e., terms containing y should remain with dy and terms containing x should remain with dx .

Example: Solve the differential equation: $x(1 + y^2)dx + y(4 + x^2)dy = 0$

Solution:

$$x(1+y^2)dx + y(4+x^2)dy = 0$$

$$\Rightarrow \frac{x}{4+x^2}dx + \frac{y}{1+y^2}dy = 0$$

$$\Rightarrow \frac{1}{2} \cdot \left(\frac{2x}{4+x^2} \right) dx + \frac{1}{2} \cdot \left(\frac{2y}{1+y^2} \right) dy = 0$$

$$\Rightarrow \int \frac{2x}{4+x^2} dx = - \int \frac{2y}{1+y^2} dy$$

$$\Rightarrow \log(4+x^2) = -\log(1+y^2) + \log C$$

$$\Rightarrow \log(4+x^2)(1+y^2) = \log C$$

$$\Rightarrow (4+x^2)(1+y^2) = C$$

This is the required solution of the given differential equation.

- **Homogeneous differential equation:**

A differential equation which can be expressed as $\frac{dy}{dx} = f(x, y)$ or $\frac{dx}{dy} = g(x, y)$, where $f(x, y)$ and $g(x, y)$ are homogenous functions of degree zero is called a homogenous differential equation. To solve such an equation, we have to substitute $y = vx$ in the given differential equation and then solve it by variable separable method.

Example: Solve the differential equation: $xydy - (x^2 - 3y^2)dx = 0$

Solution:

$$xydy - (x^2 - 3y^2)dx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 - 3y^2}{xy} = F(x, y), \quad \dots (1)$$

Now,

$$\begin{aligned} F(\lambda x, \lambda y) &= \frac{\lambda^2 x^2 - 3\lambda^2 y^2}{\lambda^2 xy} = \frac{x^2 - 3y^2}{xy} \\ &= \lambda^0 f(x, y) \end{aligned}$$

F is homogeneous function of degree zero. Therefore, the given differential equation is a homogeneous differential equation.

Let $y = vx$

$$\therefore \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Now, equation (1) becomes

$$\begin{aligned}
v + x \frac{dv}{dx} &= \frac{x^2 - 3v^2 x^2}{vx^2} = \frac{1 - 3v^2}{v} \\
\Rightarrow x \frac{dv}{dx} &= \frac{1 - 3v^2}{v} - v = \frac{1 - 4v^2}{v} \\
\Rightarrow \int \frac{v}{1 - 4v^2} dv &= \int \frac{dx}{x} \\
\Rightarrow -\frac{1}{8} \int \frac{-8v}{1 - 4v^2} dv &= \int \frac{dx}{x} \\
\Rightarrow -\frac{1}{8} \log(1 - 4v^2) &= \log(x) - \log C_1 \\
\Rightarrow \log \left[x(1 - 4v^2)^{\frac{1}{8}} \right] &= \log C_1 \\
\Rightarrow x(1 - 4v^2)^{\frac{1}{8}} &= C_1 \\
\Rightarrow x^8(1 - 4v^2) &= C_1^8 = C \text{ (say)} \\
\Rightarrow x^8 \left(1 - 4 \times \frac{y^2}{x^2} \right) &= C \\
\Rightarrow x^6(x^2 - 4y^2) &= C
\end{aligned}$$

This is the required solution of the given differential equation.

- **Linear differential equation:**

- A differential equation which can be expressed in the form of $\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of x only, is called a first order linear differential equation.

In this case, we find integrating factor (I.F.) by using the formula:

$$I.F. = e^{\int P dx}$$

Then, the solution of the differential equation is given by,

$$y(I.F.) = \int (Q \times I.F.) dx + C$$

- A linear differential equation can also be of the form $\frac{dx}{dy} + P_1x = Q$, where P_1 and Q_1 are constants or functions of y only.

In this case, $I.F. = e^{\int P_1 dy}$

And the solution of the differential equation is given by,

$$x(I.F.) = \int (Q_1 \times I.F.) dy + C$$

Example: Find the solution of the differential equation $\sin y dx = \cos y (\sin y - x) dy$, satisfying the condition that $x = 5$ when $y = \frac{\pi}{2}$.

Solution:

We have,

$$\sin y dx = \cos y (\sin y - x) dy$$

$$\Rightarrow \frac{dx}{dy} = \frac{\cos y (\sin y - x)}{\sin y} = \cos y - x \cot y$$

$$\Rightarrow \frac{dx}{dy} + x \cot y = \cos y$$

This is a linear differential equation of the form $\frac{dx}{dy} + P_1x = Q$ where $P_1 = \cot y$ and $Q_1 = \cos y$

$$\text{Now, } I.F. = e^{\int P_1 dy} = e^{\int \cot y dy} = e^{\log \sin y} = \sin y$$

Therefore, the general solution of the given differential equation is

$$x \sin y = \int \cos y \times \sin y \, dy + C$$

$$\Rightarrow x \sin y = \frac{1}{2} \int \sin 2y \, dy + C$$

$$\Rightarrow x \sin y = -\frac{1}{4} \cos 2y + C$$

Substituting $y = \frac{\pi}{2}$ and $x = 5$ in this equation, we obtain

$$5 \sin \left(\frac{\pi}{2} \right) = -\frac{1}{4} \cos \left(2 \times \frac{\pi}{2} \right) + C$$

$$5 \sin \left(\frac{\pi}{2} \right) = -\frac{1}{4} \cos \pi + C$$

$$5 \times 1 = -\frac{1}{4} \times (-1) + C$$

$$\Rightarrow C = \frac{19}{4}$$

Therefore, the required solution is

$$x \sin y = -\frac{1}{4} \cos 2y + \frac{19}{4}$$

$$\Rightarrow x \sin y + \frac{1}{4} \cos 2y = \frac{19}{4}$$

$$\Rightarrow 4x \sin y + \cos 2y = 19$$