

# Continuity and Differentiability

- Suppose  $f$  is a real function on a subset of the real numbers and  $c$  be a point in the domain of  $f$ . Then,  $f$  is continuous at  $c$ , if  $\lim_{x \rightarrow c} f(x) = f(c)$

More elaborately, we can say that  $f$  is continuous at  $c$ , if

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c)$$

- If  $f$  is not continuous at  $c$ , then we say that  $f$  is discontinuous at  $c$  and  $c$  is called the point of discontinuity.
- A real function  $f$  is said to be continuous, if it is continuous at every point in the domain of  $f$ .

- If  $f$  and  $g$  are two continuous real functions, then

- $(f + g), (f - g), f \cdot g$  are continuous

- $\frac{f}{g}$  is continuous provided  $g$  assumes non zero value.

- If  $f$  and  $g$  are two continuous functions, then  $f \circ g$  is also continuous.

- Suppose  $f$  is a real function and  $c$  is a point in its domain. Then, the derivative of

$f$  at  $c$  is defined by, 
$$f'(c) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

- Derivative of a function  $f(x)$ , denoted by  $\frac{d}{dx}(f(x))$  or  $f'(x)$ , is defined by

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

## Example:

Find derivative of  $\sin 2x$ .

## Solution:

Let  $f(x) = \sin 2x$

$$\begin{aligned}\therefore f'(x) &= \lim_{h \rightarrow 0} \frac{\sin 2(x+h) - \sin 2x}{h} \\ &= \lim_{h \rightarrow 0} \frac{2\cos(2x+h) \cdot \sin h}{h} \\ &= 2 \lim_{h \rightarrow 0} \cos(2x+h) \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\ &= 2 \times \cos 2x \times 1 \\ &= 2\cos 2x\end{aligned}$$

- For two functions  $f$  and  $g$ , the rules of algebra of derivatives are as follows:

- $(f + g)' = f' + g'$
- $(f - g)' = f' - g'$
- $(fg)' = f'g' \quad \text{[Leibnitz or product rule]}$

- $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ , where  $g \neq 0$  [Quotient rule]

- Every differentiable function is continuous, but the converse is not true.

**Example:**

$f(x) = |x|$  is continuous at all points on real line, but it is not differentiable at  $x = 0$ .

Since L.H.S  $= \lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \frac{-h}{h} = -1$

$= \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \frac{h}{h} = 1$

R.H.S

$\therefore \text{L.H.S} \neq \text{R.H.S.}$

Therefore,  $f'(x)$  does not exist at  $x = 0$ ; i.e.,  $f$  is not differentiable at  $x = 0$ .

The derivatives of some useful functions are as follows:

- $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
- $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
- $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$

$$\circ \frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$$

- **Chain rule:** This rule is used to find the derivative of a composite function. Let

$f = v \circ u$ . Suppose  $t = u(x)$ ; and if both  $\frac{dt}{dx}$  and  $\frac{dv}{dt}$  exist, then  $\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$

Similarly, if  $f = (w \circ u) \circ v$ , and if  $t = v(x)$ ,  $s = u(t)$ , then

$$\frac{df}{dx} = \frac{d(w \circ u)}{dt} \cdot \frac{dt}{dx} = \frac{dw}{ds} \cdot \frac{ds}{dt} \cdot \frac{dt}{dx}$$

**Example:** Find the derivative of  $\sin^2(\log x + \cos^2 x)$ .

**Solution:**

$$\begin{aligned} \frac{d}{dx} \left[ \sin^2(\log x + \cos^2 x) \right] &= 2\sin(\log x + \cos^2 x) \times \frac{d}{dx} \left[ \sin(\log x + \cos^2 x) \right] \\ &= 2\sin(\log x + \cos^2 x) \cdot \cos(\log x + \cos^2 x) \times \frac{d}{dx} (\log x + \cos^2 x) \\ &= \sin 2(\log x + \cos^2 x) \cdot \left[ \frac{1}{x} + 2\cos x \times \frac{d}{dx} (\cos x) \right] \\ &= \sin(\log x^2 + 2\cos^2 x) \times \left( \frac{1}{x} - 2\sin x \cos x \right) \\ &= \left( \frac{1}{x} - \sin 2x \right) \sin(\log x^2 + 2\cos^2 x) \end{aligned}$$

The derivatives of exponential functions are as follows:

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(e^{ax}) = ae^{ax}$

- **Mean value theorem:**

If  $f: [a, b] \rightarrow \mathbf{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there

exists some  $c \in (a, b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$ .

**Example:** Verify Mean Value Theorem for the function:

$f(x) = 2x^2 - 17x + 30$  in the interval  $\left[\frac{5}{2}, 6\right]$ .

**Solution:**

$$f(x) = 2x^2 - 17x + 30$$

$$\therefore f'(x) = 4x - 17$$

The function  $f(x)$  being a polynomial, is continuous on  $\left[\frac{5}{2}, 6\right]$  and is differentiable on  $\left(\frac{5}{2}, 6\right)$ .

$$\text{Also, } f\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right)^2 - 17\left(\frac{5}{2}\right) + 30 = 0$$

$$\text{and, } f(6) = 2(6)^2 - 17 \times 6 + 30 = 0$$

$$\therefore f\left(\frac{5}{2}\right) = f(6)$$

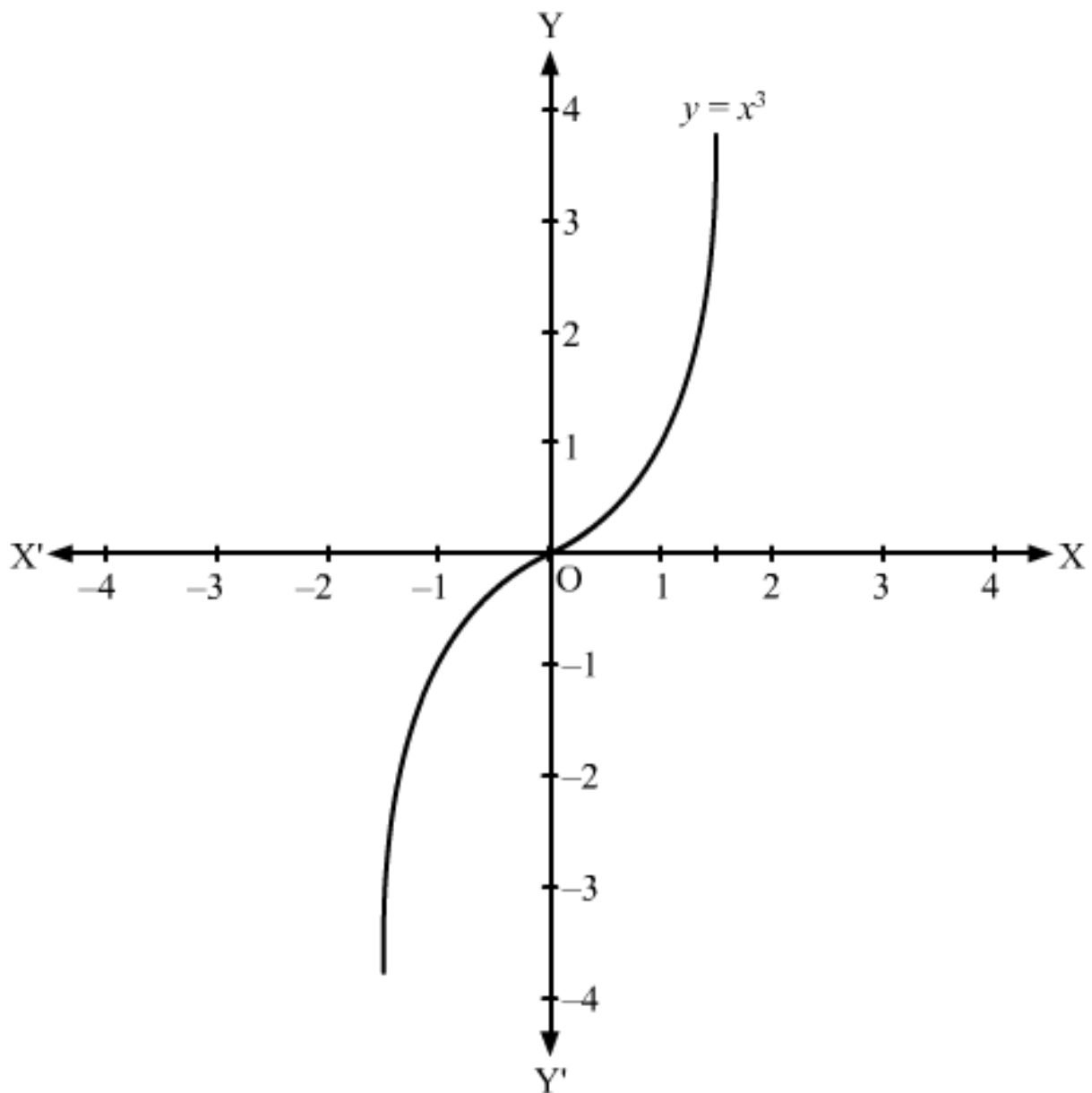
$$\frac{f(6) - f\left(\frac{5}{2}\right)}{6 - \frac{5}{2}} = 0$$

Now,

According to Mean Value Theorem (MVT), there exists  $c \in \left(\frac{5}{2}, 6\right)$  such that  $f'(c) = 0$ .

$$\therefore 4c - 17 = 0$$

$$\Rightarrow c = \frac{17}{4} \in \left(\frac{5}{2}, 6\right)$$



Therefore, M.V.T is verified.

- Derivative of a function  $f(x) = [u(x)]^{v(x)}$  can be calculated by taking logarithm on both the sides, i.e.  $\log f(x) = v(x)\log [u(x)]$ , and then differentiating both sides with respect to  $x$ .

**Example:** If  $y = x^{x^x}$ , find  $\frac{dy}{dx}$

**Solution:**

Let If  $y = x^{x^x} = x^{x^y}$

$$\therefore \log y = y \log x$$

$$\Rightarrow \frac{d}{dx}(\log y) = \frac{d}{dx}(y \log x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{dy}{dx} \log x + \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} \left[ \frac{1}{y} - \log x \right] = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{y}{x}}{\frac{1}{y} - \log x} = \frac{y^2}{x - xy \log x}$$

- If the variables  $x$  and  $y$  are expressed in the form of  $x = f(t)$  and  $y = g(t)$ , then they are said to be in parametric form. In this case,  $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{g'(t)}{f'(t)}$ , provided  $f'(t) \neq 0$

- If  $y = f(x)$ , then  $\frac{dy}{dx} = f'(x)$  and  $\frac{d^2y}{dx^2}$  or  $f''(x) = \frac{d}{dx} \left( \frac{dy}{dx} \right)$

Here,  $f''(x)$  or  $\frac{d^2y}{dx^2}$  is called the second order derivative of  $y$  with respect to  $x$ .

- **Rolle's Theorem:**

If  $f: [a, b] \rightarrow \mathbf{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$  such that  $f(a) = f(b)$ , where  $a$  and  $b$  are some real numbers, then there exists some  $c \in (a, b)$  such that  $f'(c) = 0$

**Example:** Verify Rolle's Theorem for the function:

$f(x) = 2x^2 - 17x + 30$  in the interval  $\left[\frac{5}{2}, 6\right]$ .

**Solution:**

$$f(x) = 2x^2 - 17x + 30$$

$$\therefore f'(x) = 4x - 17$$

The function  $f(x)$  being a polynomial, is continuous on  $\left[\frac{5}{2}, 6\right]$  and is differentiable on  $\left(\frac{5}{2}, 6\right)$ .

$$\text{Also, } f\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right)^2 - 17\left(\frac{5}{2}\right) + 30 = 0$$

$$\text{And, } f(6) = 2(6)^2 - 17 \times 6 + 30 = 0$$

$$\therefore f\left(\frac{5}{2}\right) = f(6)$$

Therefore, we can apply Rolle's Theorem for  $f(x)$ .

According to this theorem, there exists  $c \in \left(\frac{5}{2}, 6\right)$  such that  $f'(c) = 0$

We have  $f'(x) = 4x - 17$

$$\therefore f'(c) = 0$$

$$\Rightarrow 4c - 17 = 0$$

$$\Rightarrow c = \frac{17}{4} \in \left(\frac{5}{2}, 6\right)$$

Therefore, Rolle's Theorem is verified.

- **Mean value theorem:**

If  $f: [a, b] \rightarrow \mathbf{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there

exists some  $c \in (a, b)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$

**Example:** Verify Mean Value Theorem for the function:

$f(x) = 2x^2 - 17x + 30$  in the interval  $\left[\frac{5}{2}, 6\right]$ .

**Solution:**

$$f(x) = 2x^2 - 17x + 30$$

$$\therefore f'(x) = 4x - 17$$

The function  $f(x)$  being a polynomial, is continuous on  $\left[\frac{5}{2}, 6\right]$  and is differentiable on  $\left(\frac{5}{2}, 6\right)$ .

$$\text{Also, } f\left(\frac{5}{2}\right) = 2\left(\frac{5}{2}\right)^2 - 17\left(\frac{5}{2}\right) + 30 = 0$$

$$\text{And, } f(6) = 2(6)^2 - 17 \times 6 + 30 = 0$$

$$\therefore f\left(\frac{5}{2}\right) = f(6)$$

$$\text{Now, } \frac{f(6) - f\left(\frac{5}{2}\right)}{6 - \frac{5}{2}} = 0$$

According to Mean Value Theorem (MVT), there exists  $c \in \left(\frac{5}{2}, 6\right)$  such that  $f'(c) = 0$

$$\therefore 4c - 17 = 0$$

$$\Rightarrow c = \frac{17}{4} \in \left(\frac{5}{2}, 6\right)$$

Therefore, M.V.T is verified.