

Definite Integrals

Q.1. Evaluate : $\int_0^9 f(x) dx$, where $f(x)$ is defined by $f(x) = \sin x$, $x \in [0, \pi/2]$; $f(x) = 1$, $x \in [\pi/2, 5]$ and $f(x) = e^{x-5}$, $x \in [5, 9]$.

Solution : 1

$$\begin{aligned}\int_0^9 f(x) dx &= \int_0^{\pi/2} \sin x dx + \int_{\pi/2}^5 1 dx + \int_5^9 e^{x-5} dx \\&= [-\cos x]_0^{\pi/2} + [x]_{\pi/2}^5 + [e^{x-5}]_5^9 \\&= [1] + [5 - \pi/2] + [e^4 - 1] \\&= [e^4 + 5 - \pi/2]\end{aligned}$$

Q.2. Evaluate : $\int_0^{\pi/4} \log(1 + \tan x) dx$.

Solution : 2

$$I = \int_0^{\pi/4} \log(1 + \tan x) dx = \int_0^{\pi/4} \log \{1 + \tan(\pi/4 - x)\} dx$$

$$[\text{As, } \int_0^a f(x) dx = \int_0^a f(a-x) dx]$$

$$\text{Therefore, } I = \int_0^{\pi/4} \log [1 + (\tan^{\pi/4} - \tan x) / (1 + \tan^{\pi/4} \cdot \tan x)] dx$$

$$= \int_0^{\pi/4} \log [1 + (1 - \tan x) / (1 + \tan x)] dx$$

$$= \int_0^{\pi/4} \log [2 / (1 + \tan x)] dx$$

$$= \int_0^{\pi/4} \log 2 dx - \int_0^{\pi/4} \log (1 + \tan x) dx$$

$$= \int_0^{\pi/4} \log 2 dx - I$$

$$\text{Or, } 2I = \int_0^{\pi/4} \log 2 dx = [x \log 2]_0^{\pi/4}$$

$$\text{Or, } 2I = \pi/4 \log 2 \text{ Therefore, } I = \pi/8 \log 2.$$

Q.3. Prove that $\int_0^{2\pi} x \cos x / (1 + \cos x) dx = 2\pi^2$.

Solution : 3

$$I = \int_0^{2\pi} x \cos x / (1 + \cos x) dx \text{ ----- (1)}$$

$$= \int_0^{2\pi} (2\pi - x) \cos (2\pi - x) / [1 + \cos (2\pi - x)] dx$$

$$= \int_0^{2\pi} (2\pi - x) \cos x / (1 + \cos x) dx \text{ ----- (2)}$$

Adding (1) and (2) we get,

$$I + I = \int_0^{2\pi} (x \cos x + 2\pi \cos x - x \cos x) / (1 + \cos x) dx$$

$$2I = \int_0^{2\pi} 2\pi \cos x / (1 + \cos x) dx$$

$$= 2\pi \int_0^{2\pi} \cos x (1 - \cos x) / [(1 + \cos x)(1 - \cos x)] dx$$

$$= 2\pi \int_0^{2\pi} (\cos x - \cos^2 x) / \sin^2 x dx$$

$$= 2\pi \int_0^{2\pi} [(\cos x / \sin^2 x) - (\cos^2 x / \sin^2 x)] dx$$

$$= 2\pi \int_0^{2\pi} (\cot x \operatorname{cosec} x - \cot^2 x) dx$$

$$= 2\pi \int_0^{2\pi} (\cot x \operatorname{cosec} x - \operatorname{cosec}^2 x + 1) dx$$

$$= 2\pi [-\operatorname{cosec} x + \cot x + x]_0^{2\pi}$$

$$= 2\pi [-\operatorname{cosec} 2\pi + \cot 2\pi + 2\pi - (-\operatorname{cosec} 0 + \cot 0 + 0)]$$

$$= 2\pi[2\pi] = 4\pi^2$$

Therefore, $I = 2\pi^2$.

Q.4. Prove that $\int_0^{\pi/2} [3 \sin \theta + 4 \cos \theta] / [\sin \theta + \cos \theta] d\theta = 7\pi/4$.

Solution : 4

$$\begin{aligned}
\text{Let } I &= \int_0^{\pi/2} [3\sin\theta + 4\cos\theta]/[\sin\theta + \cos\theta] d\theta \text{ ----- (1)} \\
&= \int_0^{\pi/2} [3\sin(\pi/2 - \theta) + 4\cos(\pi/2 - \theta)]/[\sin(\pi/2 - \theta) + \cos(\pi - \theta)] d\theta \\
&\text{[As, } \int_a^b f(x) dx \\
&= \int_a^b f(a + b - x) dx \text{]} \\
&= \int_0^{\pi/2} [3\cos\theta + 4\sin\theta]/[\cos\theta + \sin\theta] d\theta \text{ ----- (2)} \\
\text{Adding (1) and (2) , } 2I &= \int_0^{\pi/2} 7[\sin\theta + \cos\theta]/[\sin\theta + \cos\theta] d\theta \\
&= 7 \int_0^{\pi/2} d\theta = 7[\theta]_0^{\pi/2} = 7 \times (\pi/2) \\
\text{Or, } I &= 7\pi/2 .
\end{aligned}$$

Q.5. Evaluate : $\int_0^{\pi/4} (\tan x + \cot x) - 1 dx$.

Solution : 5

$$\begin{aligned}
\text{We have } &\int_0^{\pi/4} (\tan x + \cot x) - 1 dx \\
&= \int_0^{\pi/4} [1/(\tan x + \cot x)] dx \\
&= \int_0^{\pi/4} [1/\{(\sin^2 x + \cos^2 x)/\sin x \cos x\}] dx \\
&= 1/2 \int_0^{\pi/4} \sin 2x dx = 1/4 [-\cos 2x]_0^{\pi/4} \\
&= 1/4 [-\cos(2 \times \pi/4) + \cos(2 \times 0)] \\
&= 1/4 [-\cos \pi/2 + \cos 0^\circ] \\
&= 1/4 [0 + 1] = 1/4 .
\end{aligned}$$

Q.6. Evaluate : $\int_0^{\pi/4} [2 \cos 2x/(1 + \sin 2x)] dx$.

Solution : 6

$$\text{Let } I = \int_0^{\pi/4} [2 \cos 2x / (1 + \sin 2x)] dx$$

$$\text{Put } 1 + \sin 2x = t \Rightarrow 2 \cos 2x dx = dt ; \text{ when } x = 0, t = 1 + 0 = 1$$

$$\text{and when } x = \pi/4, t = 1 + \sin \pi/2 = 1 + 1 = 2.$$

$$\begin{aligned} \text{Therefore, } I &= \int_1^2 dt/t = [\log t]_1^2 = \log 2 - \log 1 \\ &= \log 2 - 0 = \log 2. \end{aligned}$$

Q.7. Evaluate : $\int_0^1 x \tan^{-1} x dx$.

Solution : 7

$$\text{Let } I = \int_0^1 x \tan^{-1} x dx$$

$$= [\tan^{-1} x \cdot \int x dx - \int \{1/(1 + x^2)\} \cdot 1/2 x^2 dx]_0^1$$

$$= [\tan^{-1} x \cdot (1/2)x^2 - 1/2 \int \{x^2/(1 + x^2)\} dx]_0^1$$

$$= [1/2 x^2 \tan^{-1} x - 1/2 \int dx + 1/2 \int dx/(1 + x^2)]_0^1$$

$$= [1/2 x^2 \tan^{-1} x - 1/2 x + 1/2 \tan^{-1} x]_0^1$$

$$= 1/2 [(1 + x^2) \tan^{-1} x - x]_0^1$$

$$= 1/2 [(1 + 1) \tan^{-1} (1) - 1 - (1 + 0) \tan^{-1} (0) - 0]$$

$$= 1/2 [2 \tan^{-1} (1) - 1] = 1/2 [2 \times (\pi/4) - 1]$$

$$= \pi/4 - 1/2 .$$

Q.8. Evaluate : $\int_0^{\pi/2} \sin 2x \log \tan x dx$.

Solution : 8

$$\text{We are given, } I = \int_0^{\pi/2} \sin 2x \log \tan x dx \text{ ----- (1) Or,}$$

$$\begin{aligned}
 I &= \int_0^{\pi/2} \log \tan (\pi/2 - x) \sin 2(\pi/2 - x) dx \\
 &= \int_0^{\pi/2} \log \cot x \sin (\pi - 2x) dx \\
 &= \int_0^{\pi/2} \log \cot x \sin 2x dx \text{ ----- (2)}
 \end{aligned}$$

adding (1) and (2) we get

$$\begin{aligned}
 2 I &= \int_0^{\pi/2} \sin 2x [\log \tan x + \cot x] dx \\
 &= \int_0^{\pi/2} \sin 2x \log(\tan x \cdot \cot x) dx \\
 &= \int_0^{\pi/2} \sin 2x \log(1) dx = 0 \text{ [As, } \log(1) = 0]
 \end{aligned}$$

Therefore, $I = 0$.

Q.9. Evaluate : $\int_0^{\pi/2} \{\sqrt{(\sec x)}\} / [\sqrt{(\sec x)} + \sqrt{(\operatorname{cosec} x)}] dx$.

Solution : 9

$$\text{Let } I = \int_0^{\pi/2} [\{\sqrt{(\sec x)}\} / \{\sqrt{(\sec x)} + \sqrt{(\operatorname{cosec} x)}\}] dx \text{ ----- (1)}$$

$$\begin{aligned}
 \text{Then } I &= \int_0^{\pi/2} [\sqrt{\{\sec (\pi/2 - x)\}} / [\sqrt{\{\sec (\pi/2 - x)\}} + \sqrt{\{\operatorname{cosec} (\pi/2 - x)\}}] dx \\
 &= \int_0^{\pi/2} \{\{\sqrt{(\operatorname{cosec} x)}\} / \{\sqrt{(\operatorname{cosec} x)} + \sqrt{(\sec x)}\}\} dx \text{ ----- (2)}
 \end{aligned}$$

adding (1) and (2), we get

$$\begin{aligned}
 2I &= \int_0^{\pi/2} [\{\sqrt{(\sec x)}\} / \{\sqrt{(\sec x)} + \sqrt{(\operatorname{cosec} x)}\}] dx + \\
 &\quad \int_0^{\pi/2} [\{\sqrt{(\operatorname{cosec} x)}\} / \{\sqrt{(\operatorname{cosec} x)} + \sqrt{(\sec x)}\}] dx \\
 &= \int_0^{\pi/2} [\{\sqrt{(\sec x)} + \sqrt{(\operatorname{cosec} x)}\} / \{\sqrt{(\sec x)} + \sqrt{(\operatorname{cosec} x)}\}] dx \\
 &= \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \pi/2
 \end{aligned}$$

Therefore, $I = \pi/4$.

Q.10. Evaluate : $\int_a^b (\log x/x) dx$.

Solution : 10

We are given,

$$a \int^b (\log x/x) dx$$

$$\text{Let } I = \int (\log x)(1/x)dx$$

$$= \log x \int (1/x)dx - \int (1/x) \log x dx$$

$$= \log x \cdot \log x - I$$

$$\text{Or, } 2I = (\log x)^2$$

$$\text{Or, } I = [(\log x)^2]/2$$

$$\text{Therefore, } a \int^b (\log x/x)dx = [\{(\log x)^2\}/2]_{ab} = \{(\log b)^2\}/2 - \{(\log a)^2\}/2$$

$$= 1/2 (\log b + \log a)(\log b - \log a) = 1/2 \log (ab) \log (a/b).$$

Q.11. Evaluate : $0 \int^{1/2} \sin^{-1} x / (1 - x^2)^{3/2} dx$.

Solution : 11

$$\text{We have, } I = a \int^{1/2} \sin^{-1} x / (1 - x^2)^{3/2} dx$$

$$\text{Put } \sin^{-1} x = t \Rightarrow x = \sin t \text{ and } dx = \cos t dt$$

$$\text{Therefore, } I = a \int^{1/2} \sin^{-1} x / (1 - x^2)^{3/2} dx$$

$$= a \int^{\pi/6} (t / \cos^3 t) \cdot (\cos t) dt,$$

$$[x = 0, \sin t = 0 \Rightarrow t = 0; x = 1/2, \sin t = 1/2 \Rightarrow t = \pi/6]$$

$$= a \int^{\pi/6} (t / \cos^2 t) dt = 0 \int^{\pi/6} t \sec^2 t dt$$

$$= [t \int \sec^2 t dt - \int \{d/dt(t) \cdot \int \sec^2 t dt\}]_0^{\pi/6}$$

$$= [t \cdot \tan t - \int \tan t dt]_0^{\pi/6} = [t \cdot \tan t - \log \sec t]_0^{\pi/6}$$

$$\begin{aligned}
&= [\pi/6 \tan \pi/6 - \log \sec \pi/6] - [0 - \log \sec 0] \\
&= [\pi/6 (1/\sqrt{3}) - \log (2/\sqrt{3})] - [0 - 0] \\
&= \pi/(6\sqrt{3}) - \log (2/\sqrt{3}).
\end{aligned}$$

Q.12. Evaluate $\int_0^{\pi/2} \log(\tan x) dx$.

Solution : 12

$$\begin{aligned}
\text{We have, } I &= \int_0^{\pi/2} \log(\tan x) dx = \int_0^{\pi/2} \log[\tan(\pi/2 - x)] dx \\
&= \int_0^{\pi/2} \log(\cot x) dx
\end{aligned}$$

$$\begin{aligned}
\text{Therefore, } 2I &= \int_0^{\pi/2} [\log(\tan x) + \log(\cot x)] dx \\
&= \int_0^{\pi/2} \log(\tan x \cdot \cot x) dx = \int_0^{\pi/2} \log 1 dx = 0
\end{aligned}$$

Therefore, $I = 0$.