Cross Product of vectors

Q.1. Given $\vec{a} = i - 2j + k$, $\vec{b} = 2i + j + k$ and $\vec{c} = i + 2j - k$. Find : $\vec{a} \times (\vec{b} \times \vec{c})$.

Solution : 1

$a \times (b \times c) = (i - 2j + k) \times (2i + j + k) \times (i + 2j - k)$

$= (a.c)b - (a.b)c$

$= (1 - 4 - 1)(2i + j + k) - (2 - 2 + 1)(i + 2j - k)$

$= - 8i - 4j - 4k - i - 2j + k$

$= - 9i - 6j - 3k$.

Q.2. Find a unit vector perpendicular to the vectors $4i + 3j + k$ and $2i - j + 2k$. Determine the sine angle between these two vectors.

Solution : 2

Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ is given by :

$[\vec{a}\times\vec{b}]/|\vec{a}\times\vec{b}|$

Therefore, unit vector perpendicular to $4i + 3j + k$ and $2i - j + k$ is

$= [(4i + 3j + k) \times (2i - j + 2k)]/|(4i + 3j + k) \times(2i - j + 2k)|$

$= [7i - 6j - 10k]/\sqrt{(47 + 36 + 100)}$

$= (7/\sqrt{185})I - (6/\sqrt{185})j - 10/\sqrt{185}k$.

Q.3. The vectors $- 2i + 4j + 4k$ and $- 4i - 2k$ represent the diagonals BD and AC of a parallelogram ABCD. Find the area of the parallelogram.

Solution : 3

Area of parallelogram = 1/2 $|\vec{BD} \times \vec{AC}|$ = 1/2 $|(-2i + 4j + 4k)\times(-4i - 2k)|$

$| \; i \; j \; k |$

= 1/2 $|-2 \; 4 \; 4|$

$|-4 \; 0 \; -2|$

= 1/2 $|i(-8) - j(4 + 16) + k(16)|$

= 1/2 $|-8i - 20j + 16k|$

= 1/2 $\sqrt{(8^2 + 20^2 + 16^2)}$

= 1/2 $\sqrt{(720)}$ = 1/2×12$\sqrt{5}$ = 6$\sqrt{5}$.

Q.4. Find the unit vector perpendicular to the two vectors : $\vec{i} + \vec{j} - \vec{k}$ and $2\vec{i} + 3\vec{j} + \vec{k}$.

Solution : 4

Let $\vec{a} = \vec{i} + \vec{j} - \vec{k}$ and $\vec{b} = 2\vec{i} + 3\vec{j} + \vec{k}$

Unit vector perpendicular to $\vec{a}$ and $\vec{b}$ = $[\vec{a} \times \vec{b}]/|\vec{a} \times \vec{b}|$

[$\vec{a} \times \vec{b} = \vec{i}(2 + 3) - \vec{j}(1 + 2) + \vec{k}(3 - 4) = 5\vec{i} - 3\vec{j} - \vec{k}$

and $|\vec{a} \times \vec{b}| = |5\vec{i} - 3\vec{j} - \vec{k}| = \sqrt{(25 + 9 + 1)} = \sqrt{(35)}$]

Therefore, unit vector perpendicular to $\vec{a}$ and $\vec{b}$

= $[5\vec{i} - 3\vec{j} - \vec{k}]/\sqrt{(35)}$.

Q.5. Find the area of the parallelogram ABCD whose diagonals AC and BD are represented by the vectors 3i + j − 2k and i − 3j − 4k respectively.

Solution : 5

We are given , $\vec{AC}$ = 3i + j − 2k and $\vec{BD}$ = i − 3j − 4k.

Area of parallelogram = 1/2 $|\vec{AC} \times \vec{BD}|$

$\vec{AC} \times \vec{BD}$ = (3i + j − 2k)×(i − 3j − 4k)

$= i(4 - 6) - j\,(12 + 2) + k\,(-9 - 1)$

$= -2i - 14j - 10k$

$|\vec{AC} \times \vec{BD}| = \sqrt{(4 + 196 + 100)} = \sqrt{(300)} = 10\sqrt{3}$

Area of parallelogram = $1/2\ |\vec{AC} \times \vec{BD}| = 1/2 \times 10\sqrt{3} = 5\sqrt{3}$.

Q.6. Find the area of a parallelogram whose diagonals are determined by the vectors : $\vec{a} = 3i + j - 2k$ and $\vec{b} = i - 3j + 4k$.

Solution : 6

Area = $1/2|\ d1 \times d2|$

$|\ i\ j\ k\ |$

$= 1/2\ |\ 3\ 1\ -2\ |$

$|\ 1\ -3\ 4\ |$

$= 1/2\ |\{i(4 - 6) - j\,(12 + 2) + k\,(-9 - 1)|$

$= 1/2\ |(-2i - 14j - 10k)|$

$= 1/2\ \sqrt{(4 + 196 + 100)}$

$= 1/2\ \sqrt{(300)} = 5\sqrt{3}$ sq. units.

Q.7. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ represents the position vectors of the points with co-ordinates (2, – 10, 2), (3, 1, 2) and (2, 1, 3) respectively, find the value of $\vec{a} \times (\vec{b} \times \vec{c})$.

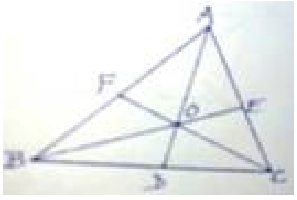
Solution : 7

$\vec{a} \times (\vec{b} \times \vec{c}) = (2i - 10j + 2k) \times \{(3i + j + 2k) \times (2i + j + 3k)\}$

$= (2i - 10j + 2k) \times (i - 5j + k) = 0.$

Q.8. Using vectors, show that the medians of a triangle meet at a point.

Solution : 8



Fig

Let D, E and F be mid-points of sides BC, CA and AB of Δ ABC and O the point of intersection of medians AD and BE. Let position vectors of A, B and C be $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively.

$\vec{OD} = \vec{OB} - \vec{DB}$

$= \vec{OB} - 1/2\vec{CB}$

$= \vec{OB} - 1/2(\vec{OB} - \vec{OC})$

$= 1/2(\vec{OB} + \vec{OC})$

$= 1/2(\vec{b} + \vec{c})$

Similarly, $\vec{OE} = 1/2(\vec{a} + \vec{c})$

and $\vec{OF} = 1/2(\vec{a} + \vec{b})$

$\vec{OA}$ and $\vec{OD}$ being in opposite direction,

$\vec{OA} \times \vec{OD} = 0$

Or, $\vec{a} \times 1/2(\vec{b} + \vec{c}) = 0$

Or, $1/2[\vec{a} \times (\vec{b} + \vec{c})] = 0$ -------------------- (1)

Similarly, $1/2[\vec{b} \times (\vec{a} + \vec{c})] = 0$ -------------------- (2)

Adding (1) and (2), we get

$1/2[\vec{a} \times (\vec{b} + \vec{c})] + 1/2[\vec{b} \times (\vec{a} + \vec{c})] = 0$

Or, $1/2[\vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{a} + \vec{b} \times \vec{c}] = 0$

Or, $\vec{a} \times \vec{c} + \vec{b} \times \vec{c} = 0$ $[\vec{a} \times \vec{b} = - \vec{b} \times \vec{a}]$

Or, $(\vec{a} + \vec{b}) \times \vec{c} = 0$

Or, $1/2(\vec{a} + \vec{b}) \times \vec{c} = 0$

Or, $\vec{OC} \times \vec{OF} = 0$

Thus we see that $\vec{OC}$ and $\vec{OF}$ are in opposite in direction i.e., median CF also passes through the point 'O' which is the intersection point of median AD and BE.
Therefore, medians of a triangle meet at a point.

Q.9. Given that $\vec{a} = i - 2j + k$, $\vec{b} = 2i + j + k$ and $\vec{c} = i + 2j - k$. Find the vector $\vec{a} \times (\vec{b} \times \vec{c})$.

Solution : 9

We have, $\vec{b} \times \vec{c} = (2i + j + k) \times (i + 2j - k)$

$$= \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix}$$

$= i [(1)(-1) - (1)(2)] - j [(2)(-1) - (1)(1)] + [(2)(2) - (1)(1)]$

$= -3i + 3j + 3k$

Therefore, $\vec{a} \times (\vec{b} \times \vec{c}) = (i - 2j + k) \times (-3i + 3j + 3k)$

$$= \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ -3 & 3 & 3 \end{vmatrix}$$

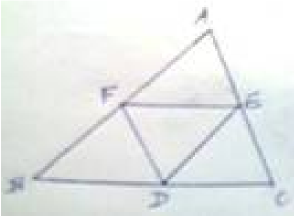
$= i[(-2)(3) - (1)(3)] - j [(1)(3) - (1)(-3)] + k [(1)(3) - (-2)(-3)]$

$= i(-9) - j(6) + k(-3)$

$= -9i - 6j - 3k$

Q.10. If D, E, F are the mid-points of the sides BC, CA, AB respectively of a triangle ABC. Show that the area of triangle DEF = 1/4[area of Δ ABC].

Solution : 10



Fig

Let A be the origin and AB and AC represents $\vec{b}$ and $\vec{c}$.

Then, $\vec{DE} = -(1/2)\vec{b}$; $\vec{DF} = -(1/2)\vec{c}$.

Vector area of ΔDEF = $1/2\vec{DE} \times \vec{DF}$

$= 1/2[-1/2\vec{b} \times -1/2\vec{c}]$

$= 1/8(\vec{b} \times \vec{c})$

$= 1/4(1/2\ \vec{b} \times \vec{c})$

$= 1/4[1/2\ \vec{AB} \times \vec{AC}]$

= 1/4 [Vector area of ΔABC]

Q.11. Show that : $i \times(\vec{a} \times i) + j \times(\vec{a} \times j) + k \times(\vec{a} \times k) = 2\vec{a}$ where, $\vec{a} = a_1i + a_2j + a_3k$.

Solution : 11

We have, $i \times(\vec{a} \times i) + j \times(\vec{a} \times j) + k \times(\vec{a} \times k)$

$= [(i.i)\vec{a} - (i.\vec{a})i] + [(j.j)\vec{a} - (j.\vec{a})j] + [(k.k)\vec{a} - (k.\vec{a})k]$

$= \vec{a} - (i.\vec{a})i + \vec{a} - (j.\vec{a})j + \vec{a} - (k.\vec{a})k$

$=3\vec{a} - [(i.\vec{a})i + (j.\vec{a})j + (k.\vec{a})k]$

$= 3\vec{a} - [(a_1i) + (a_2j) + (a_3k)]$

$= 3\vec{a} - \vec{a}$

$= 2\vec{a}$.