

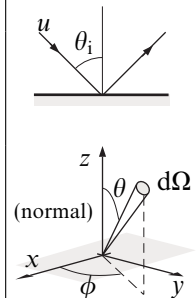
7.8 Waves in and out of media

Waves in lossless media

Electric field	$\nabla^2 \mathbf{E} = \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$	(7.193)	$\mathbf{E}$ electric field μ permeability ($=\mu_0\mu_r$) ϵ permittivity ($=\epsilon_0\epsilon_r$)
Magnetic field	$\nabla^2 \mathbf{B} = \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2}$	(7.194)	$\mathbf{B}$ magnetic flux density t time
Refractive index	$\eta = \sqrt{\epsilon_r\mu_r}$	(7.195)	
Wave speed	$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{\eta}$	(7.196)	v wave phase speed η refractive index c speed of light
Impedance of free space	$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \simeq 376.7 \Omega$	(7.197)	Z_0 impedance of free space
Wave impedance	$Z = \frac{E}{H} = Z_0 \sqrt{\frac{\mu_r}{\epsilon_r}}$	(7.198)	Z wave impedance H magnetic field strength

Radiation pressure^a

Radiation momentum density	$\mathbf{G} = \frac{\mathbf{N}}{c^2}$	(7.199)	$\mathbf{G}$ momentum density $\mathbf{N}$ Poynting vector c speed of light
Isotropic radiation	$p_n = \frac{1}{3}u(1+R)$	(7.200)	p_n normal pressure u incident radiation energy density R (power) reflectance coefficient
Specular reflection	$p_n = u(1+R)\cos^2\theta_i$	(7.201)	p_t tangential pressure
	$p_t = u(1-R)\sin\theta_i\cos\theta_i$	(7.202)	θ_i angle of incidence
From an extended source ^b	$p_n = \frac{1+R}{c} \iint I_v(\theta, \phi) \cos^2\theta \, d\Omega \, dv$	(7.203)	I_v specific intensity v frequency Ω solid angle θ angle between $d\Omega$ and normal to plane
From a point source, ^c luminosity L	$p_n = \frac{L(1+R)}{4\pi r^2 c}$	(7.204)	L source luminosity (i.e., radiant power) r distance from source

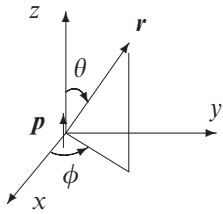


^aOn an opaque surface.

^bIn spherical polar coordinates. See page 120 for the meaning of specific intensity.

^cNormal to the plane.

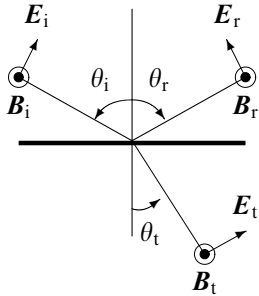
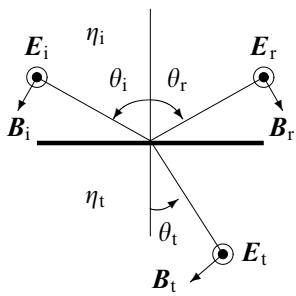
Antennas

Spherical polar geometry:			
Field from a short ($l \ll \lambda$) electric dipole in free space ^a	$E_r = \frac{1}{2\pi\epsilon_0} \left(\frac{[\dot{p}]}{r^2c} + \frac{[p]}{r^3} \right) \cos\theta \quad (7.205)$ $E_\theta = \frac{1}{4\pi\epsilon_0} \left(\frac{[\dot{p}]}{rc^2} + \frac{[\dot{p}]}{r^2c} + \frac{[p]}{r^3} \right) \sin\theta \quad (7.206)$ $B_\phi = \frac{\mu_0}{4\pi} \left(\frac{[\dot{p}]}{rc} + \frac{[p]}{r^2} \right) \sin\theta \quad (7.207)$	r distance from dipole θ angle between $\mathbf{r}$ and $\mathbf{p}$ $[p]$ retarded dipole moment $[p] = p(t-r/c)$ c speed of light	
Radiation resistance of a short electric dipole in free space	$R = \frac{\omega^2 l^2}{6\pi\epsilon_0 c^3} = \frac{2\pi Z_0}{3} \left(\frac{l}{\lambda} \right)^2 \quad (7.208)$ $\simeq 789 \left(\frac{l}{\lambda} \right)^2 \text{ ohm} \quad (7.209)$	l dipole length ($\ll \lambda$) ω angular frequency λ wavelength Z_0 impedance of free space	
Beam solid angle	$\Omega_A = \int_{4\pi} P_n(\theta, \phi) d\Omega \quad (7.210)$	Ω_A beam solid angle P_n normalised antenna power pattern $P_n(0,0) = 1$ $d\Omega$ differential solid angle	
Forward power gain	$G(0) = \frac{4\pi}{\Omega_A} \quad (7.211)$	G antenna gain	
Antenna effective area	$A_e = \frac{\lambda^2}{\Omega_A} \quad (7.212)$	A_e effective area	
Power gain of a short dipole	$G(\theta) = \frac{3}{2} \sin^2\theta \quad (7.213)$		
Beam efficiency	$\text{efficiency} = \frac{\Omega_M}{\Omega_A} \quad (7.214)$	Ω_M main lobe solid angle	
Antenna temperature ^b	$T_A = \frac{1}{\Omega_A} \int_{4\pi} T_b(\theta, \phi) P_n(\theta, \phi) d\Omega \quad (7.215)$	T_A antenna temperature T_b sky brightness temperature	

^aAll field components propagate with a further phase factor equal to $\exp i(kr - \omega t)$, where $k = 2\pi/\lambda$.

^bThe brightness temperature of a source of specific intensity I_ν is $T_b = \lambda^2 I_\nu / (2k_B)$.

Reflection, refraction, and transmission^a

parallel incidence  perpendicular incidence 		E electric field B magnetic flux density η_i refractive index on incident side η_t refractive index on transmitted side θ_i angle of incidence θ_r angle of reflection θ_t angle of refraction
Law of reflection	$\theta_i = \theta_r$	(7.216)
Snell's law ^b	$\eta_i \sin \theta_i = \eta_t \sin \theta_t$	(7.217)
Brewster's law	$\tan \theta_B = \eta_t / \eta_i$	(7.218)
		θ_B Brewster's angle of incidence for plane-polarised reflection ($r_{\parallel} = 0$)
Fresnel equations of reflection and refraction		
$r_{\parallel} = \frac{\sin 2\theta_i - \sin 2\theta_t}{\sin 2\theta_i + \sin 2\theta_t}$	(7.219)	$r_{\perp} = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$ (7.223)
$t_{\parallel} = \frac{4 \cos \theta_i \sin \theta_t}{\sin 2\theta_i + \sin 2\theta_t}$	(7.220)	$t_{\perp} = \frac{2 \cos \theta_i \sin \theta_t}{\sin(\theta_i + \theta_t)}$ (7.224)
$R_{\parallel} = r_{\parallel}^2$	(7.221)	$R_{\perp} = r_{\perp}^2$ (7.225)
$T_{\parallel} = \frac{\eta_t \cos \theta_t}{\eta_i \cos \theta_i} t_{\parallel}^2$	(7.222)	$T_{\perp} = \frac{\eta_t \cos \theta_t}{\eta_i \cos \theta_i} t_{\perp}^2$ (7.226)
Coefficients for normal incidence ^c		
$R = \frac{(\eta_i - \eta_t)^2}{(\eta_i + \eta_t)^2}$	(7.227)	$r = \frac{\eta_i - \eta_t}{\eta_i + \eta_t}$ (7.230)
$T = \frac{4\eta_i \eta_t}{(\eta_i + \eta_t)^2}$	(7.228)	$t = \frac{2\eta_i}{\eta_i + \eta_t}$ (7.231)
$R + T = 1$	(7.229)	$t - r = 1$ (7.232)
$\parallel$ electric field parallel to the plane of incidence	$\perp$ electric field perpendicular to the plane of incidence	
R (power) reflectance coefficient	r amplitude reflection coefficient	
T (power) transmittance coefficient	t amplitude transmission coefficient	

^aFor the plane boundary between lossless dielectric media. All coefficients refer to the electric field component and whether it is parallel or perpendicular to the plane of incidence. Perpendicular components are out of the paper.

^bThe incident wave suffers total internal reflection if $\frac{n_t}{n_i} \sin \theta_i > 1$.

^cI.e., $\theta_i = 0$. Use the diagram labelled "perpendicular incidence" for correct phases.

Propagation in conducting media^a

Electrical conductivity ($B = 0$)	$\sigma = n_e e \mu = \frac{n_e e^2}{m_e} \tau_c$	(7.233)	σ electrical conductivity n_e electron number density τ_c electron relaxation time μ electron mobility B magnetic flux density
Refractive index of an ohmic conductor ^b	$\eta = (1 + i) \left(\frac{\sigma}{4\pi\nu\epsilon_0} \right)^{1/2}$	(7.234)	m_e electron mass $-e$ electronic charge η refractive index ϵ_0 permittivity of free space
Skin depth in an ohmic conductor	$\delta = (\mu_0 \sigma \pi \nu)^{-1/2}$	(7.235)	ν frequency δ skin depth μ_0 permeability of free space

^aAssuming a relative permeability, μ_r , of 1.

^bTaking the wave to have an $e^{-i\omega t}$ time dependence, and the low-frequency limit ($\sigma \gg 2\pi\nu\epsilon_0$).

Electron scattering processes^a

Rayleigh scattering cross section ^b	$\sigma_R = \frac{\omega^4 \alpha^2}{6\pi\epsilon_0 c^4}$	(7.236)	σ_R Rayleigh cross section ω radiation angular frequency α particle polarisability ϵ_0 permittivity of free space
Thomson scattering cross section ^c	$\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{4\pi\epsilon_0 m_e c^2} \right)^2$ $= \frac{8\pi}{3} r_e^2 \simeq 6.652 \times 10^{-29} \text{ m}^2$	(7.237) (7.238)	σ_T Thomson cross section m_e electron (rest) mass r_e classical electron radius c speed of light
Inverse Compton scattering ^d	$P_{\text{tot}} = \frac{4}{3} \sigma_T c u_{\text{rad}} \gamma^2 \left(\frac{v^2}{c^2} \right)$	(7.239)	P_{tot} electron energy loss rate u_{rad} radiation energy density γ Lorentz factor $= [1 - (v/c)^2]^{-1/2}$ v electron speed
Compton scattering ^e	$\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos \theta)$ $h\nu' = \frac{m_e c^2}{1 - \cos \theta + (1/\epsilon)}$ $\cot \phi = (1 + \epsilon) \tan \frac{\theta}{2}$	(7.240) (7.241) (7.242)	λ, λ' incident & scattered wavelengths ν, ν' incident & scattered frequencies θ photon scattering angle $\frac{h}{m_e c}$ electron Compton wavelength $\epsilon = h\nu / (m_e c^2)$
Klein–Nishina cross section (for a free electron)	$\sigma_{\text{KN}} = \frac{\pi r_e^2}{\epsilon} \left\{ \left[1 - \frac{2(\epsilon + 1)}{\epsilon^2} \right] \ln(2\epsilon + 1) + \frac{1}{2} + \frac{4}{\epsilon} - \frac{1}{2(2\epsilon + 1)^2} \right\}$ $\simeq \sigma_T \quad (\epsilon \ll 1)$ $\simeq \frac{\pi r_e^2}{\epsilon} \left(\ln 2\epsilon + \frac{1}{2} \right) \quad (\epsilon \gg 1)$	(7.243) (7.244) (7.245)	σ_{KN} Klein–Nishina cross section

^aFor Rutherford scattering see page 72.

^bScattering by bound electrons.

^cScattering from free electrons, $\epsilon \ll 1$.

^dElectron energy loss rate due to photon scattering in the Thomson limit ($\gamma h\nu \ll m_e c^2$).

^eFrom an electron at rest.

Cherenkov radiation

Cherenkov cone angle	$\sin \theta = \frac{c}{\eta v}$	(7.246)	θ cone semi-angle c (vacuum) speed of light $\eta(\omega)$ refractive index v particle velocity
Radiated power ^a	$P_{\text{tot}} = \frac{e^2 \mu_0}{4\pi} v \int_0^{\omega_c} \left[1 - \frac{c^2}{v^2 \eta^2(\omega)} \right] \omega \, d\omega$ $\text{where } \eta(\omega) \geq \frac{c}{v} \quad \text{for } 0 < \omega < \omega_c$	(7.247)	P_{tot} total radiated power $-e$ electronic charge μ_0 free space permeability ω angular frequency ω_c cutoff frequency

^aFrom a point charge, e , travelling at speed v through a medium of refractive index $\eta(\omega)$.