2.12 Laplace transforms

Laplace transform theorems

Definition[a]	$F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t$ (2.514)	$\mathscr{L}\{\}$	Laplace transform
Convolution[b]	$F(s)\cdot G(s) = \mathscr{L}\left\{\int_0^\infty f(t-z)g(z)\,\mathrm{d}z\right\}$ (2.515) $= \mathscr{L}\{f(t) * g(t)\}$ (2.516)	$F(s)$ $G(s)$ $*$	$\mathscr{L}\{f(t)\}$ $\mathscr{L}\{g(t)\}$ convolution
Inverse[c]	$f(t) = \frac{1}{2\pi\mathbf{i}}\int_{\gamma-\mathbf{i}\infty}^{\gamma+\mathbf{i}\infty} \mathrm{e}^{st}F(s)\,\mathrm{d}s$ (2.517) $= \sum \text{residues} \quad (\text{for } t>0)$ (2.518)	γ	constant
Transform of derivative	$\mathscr{L}\left\{\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right\} = s^n\mathscr{L}\{f(t)\} - \sum_{r=0}^{n-1} s^{n-r-1}\frac{\mathrm{d}^r f(t)}{\mathrm{d}t^r}\Big\vert_{t=0}$ (2.519)	n	integer >0
Derivative of transform	$\frac{\mathrm{d}^n F(s)}{\mathrm{d}s^n} = \mathscr{L}\{(-t)^n f(t)\}$ (2.520)		
Substitution	$F(s-a) = \mathscr{L}\{\mathrm{e}^{at}f(t)\}$ (2.521)	a	constant
Translation	$\mathrm{e}^{-as}F(s) = \mathscr{L}\{u(t-a)f(t-a)\}$ (2.522) where $u(t) = \begin{cases} 0 & (t<0) \\ 1 & (t>0) \end{cases}$ (2.523)	$u(t)$	unit step function

[a] If $|\mathrm{e}^{-s_0 t}f(t)|$ is finite for sufficiently large t, the Laplace transform exists for $s > s_0$.
[b] Also known as the "faltung (or folding) theorem."
[c] Also known as the "Bromwich integral." γ is chosen so that the singularities in $F(s)$ are left of the integral line.

Laplace transform pairs

$$f(t) \Longrightarrow F(s) = \mathscr{L}\{f(t)\} = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t \tag{2.524}$$

$$\delta(t) \Longrightarrow 1 \tag{2.525}$$

$$1 \Longrightarrow 1/s \qquad (s>0) \tag{2.526}$$

$$t^n \Longrightarrow \frac{n!}{s^{n+1}} \qquad (s>0, n>-1) \tag{2.527}$$

$$t^{1/2} \Longrightarrow \sqrt{\frac{\pi}{4s^3}} \tag{2.528}$$

$$t^{-1/2} \Longrightarrow \sqrt{\frac{\pi}{s}} \tag{2.529}$$

$$\mathrm{e}^{at} \Longrightarrow \frac{1}{s-a} \qquad (s>a) \tag{2.530}$$

$$t\mathrm{e}^{at} \Longrightarrow \frac{1}{(s-a)^2} \qquad (s>a) \tag{2.531}$$

$$(1-at)\mathrm{e}^{-at} \Longrightarrow \frac{s}{(s+a)^2} \tag{2.532}$$

$$t^2\mathrm{e}^{-at} \Longrightarrow \frac{2}{(s+a)^3} \tag{2.533}$$

$$\sin at \Longrightarrow \frac{a}{s^2+a^2} \qquad (s>0) \tag{2.534}$$

$$\cos at \Longrightarrow \frac{s}{s^2+a^2} \qquad (s>0) \tag{2.535}$$

$$\sinh at \Longrightarrow \frac{a}{s^2-a^2} \qquad (s>a) \tag{2.536}$$

$$\cosh at \Longrightarrow \frac{s}{s^2-a^2} \qquad (s>a) \tag{2.537}$$

$$e^{-bt}\sin at \Longrightarrow \frac{a}{(s+b)^2+a^2} \tag{2.538}$$

$$e^{-bt}\cos at \Longrightarrow \frac{s+b}{(s+b)^2+a^2} \tag{2.539}$$

$$\mathrm{e}^{-at}f(t) \Longrightarrow F(s+a) \tag{2.540}$$