2.7 Differentiation

Derivatives (general)

Power	$\dfrac{\mathrm{d}}{\mathrm{d}x}(u^n)=nu^{n-1}\dfrac{\mathrm{d}u}{\mathrm{d}x}$	(2.292)	n	power index
Product	$\dfrac{\mathrm{d}}{\mathrm{d}x}(uv)=u\dfrac{\mathrm{d}v}{\mathrm{d}x}+v\dfrac{\mathrm{d}u}{\mathrm{d}x}$	(2.293)	u,v	functions of x
Quotient	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{u}{v}\right)=\dfrac{1}{v}\dfrac{\mathrm{d}u}{\mathrm{d}x}-\dfrac{u}{v^2}\dfrac{\mathrm{d}v}{\mathrm{d}x}$	(2.294)		
Function of a function[a]	$\dfrac{\mathrm{d}}{\mathrm{d}x}[f(u)]=\dfrac{\mathrm{d}}{\mathrm{d}u}[f(u)]\cdot\dfrac{\mathrm{d}u}{\mathrm{d}x}$	(2.295)	$f(u)$	function of $u(x)$
Leibniz theorem	$\dfrac{\mathrm{d}^n}{\mathrm{d}x^n}[uv]=\binom{n}{0}v\dfrac{\mathrm{d}^n u}{\mathrm{d}x^n}+\binom{n}{1}\dfrac{\mathrm{d}v}{\mathrm{d}x}\dfrac{\mathrm{d}^{n-1}u}{\mathrm{d}x^{n-1}}+\cdots+\binom{n}{k}\dfrac{\mathrm{d}^k v}{\mathrm{d}x^k}\dfrac{\mathrm{d}^{n-k}u}{\mathrm{d}x^{n-k}}+\cdots+\binom{n}{n}u\dfrac{\mathrm{d}^n v}{\mathrm{d}x^n}$	(2.296)	$\binom{n}{k}$	binomial coefficient
Differentiation under the integral sign	$\dfrac{\mathrm{d}}{\mathrm{d}q}\left[\int_p^q f(x)\,\mathrm{d}x\right]=f(q)\quad(p\text{ constant})$	(2.297)		
	$\dfrac{\mathrm{d}}{\mathrm{d}p}\left[\int_p^q f(x)\,\mathrm{d}x\right]=-f(p)\quad(q\text{ constant})$	(2.298)		
General integral	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\int_{u(x)}^{v(x)} f(t)\,\mathrm{d}t\right]=f(v)\dfrac{\mathrm{d}v}{\mathrm{d}x}-f(u)\dfrac{\mathrm{d}u}{\mathrm{d}x}$	(2.299)		
Logarithm	$\dfrac{\mathrm{d}}{\mathrm{d}x}(\log_b\|ax\|)=(x\ln b)^{-1}$	(2.300)	b a	log base constant
Exponential	$\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{ax})=a\mathrm{e}^{ax}$	(2.301)		
Inverse functions	$\dfrac{\mathrm{d}x}{\mathrm{d}y}=\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}$	(2.302)		
	$\dfrac{\mathrm{d}^2x}{\mathrm{d}y^2}=-\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^{-3}$	(2.303)		
	$\dfrac{\mathrm{d}^3x}{\mathrm{d}y^3}=\left[3\left(\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)^2-\dfrac{\mathrm{d}y}{\mathrm{d}x}\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}\right]\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^{-5}$	(2.304)		

[a] The "chain rule."

Trigonometric derivatives[a]

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin ax) = a\cos ax \quad (2.305)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos ax) = -a\sin ax \quad (2.306)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan ax) = a\sec^2 ax \quad (2.307)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\csc ax) = -a\csc ax\cdot\cot ax \quad (2.308)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec ax) = a\sec ax\cdot\tan ax \quad (2.309)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cot ax) = -a\csc^2 ax \quad (2.310)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin ax) = a(1-a^2x^2)^{-1/2} \quad (2.311)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arccos ax) = -a(1-a^2x^2)^{-1/2} \quad (2.312)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\arctan ax) = a(1+a^2x^2)^{-1} \quad (2.313)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arccsc}\, ax) = -\frac{a}{|ax|}(a^2x^2-1)^{-1/2} \quad (2.314)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arcsec}\, ax) = \frac{a}{|ax|}(a^2x^2-1)^{-1/2} \quad (2.315)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arccot}\, ax) = -a(a^2x^2+1)^{-1} \quad (2.316)$$

[a] a is a constant.

Hyperbolic derivatives[a]

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sinh ax) = a\cosh ax \quad (2.317)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh ax) = a\sinh ax \quad (2.318)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tanh ax) = a\,\mathrm{sech}^2 ax \quad (2.319)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{csch}\, ax) = -a\,\mathrm{csch}\, ax\cdot\coth ax \quad (2.320)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{sech}\, ax) = -a\,\mathrm{sech}\, ax\cdot\tanh ax \quad (2.321)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\coth ax) = -a\,\mathrm{csch}^2 ax \quad (2.322)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arsinh}\, ax) = a(a^2x^2+1)^{-1/2} \quad (2.323)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arcosh}\, ax) = a(a^2x^2-1)^{-1/2} \quad (2.324)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{artanh}\, ax) = a(1-a^2x^2)^{-1} \quad (2.325)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arcsch}\, ax) = -\frac{a}{|ax|}(1+a^2x^2)^{-1/2} \quad (2.326)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arsech}\, ax) = -\frac{a}{|ax|}(1-a^2x^2)^{-1/2} \quad (2.327)$$

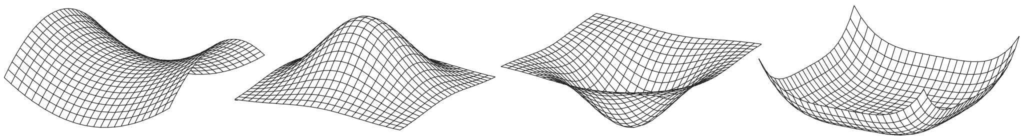
$$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{arcoth}\, ax) = a(1-a^2x^2)^{-1} \quad (2.328)$$

[a] a is a constant.

Partial derivatives

Total differential	$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z$ (2.329)	f $f(x,y,z)$
Reciprocity	$\left.\frac{\partial g}{\partial x}\right\rvert_y \left.\frac{\partial x}{\partial y}\right\rvert_g \left.\frac{\partial y}{\partial g}\right\rvert_x = -1$ (2.330)	g $g(x,y)$
Chain rule	$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}$ (2.331)	
Jacobian	$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$ (2.332)	J Jacobian u $u(x,y,z)$ v $v(x,y,z)$ w $w(x,y,z)$
Change of variable	$\int_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{V'} f(u,v,w)J\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$ (2.333)	V volume in (x,y,z) V' volume in (u,v,w) mapped to by V
Euler–Lagrange equation	if $I = \int_a^b F(x,y,y')\,\mathrm{d}x$ then $\delta I = 0$ when $\frac{\partial F}{\partial y} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial F}{\partial y'}\right)$ (2.334)	y' $\mathrm{d}y/\mathrm{d}x$ a,b fixed end points

Stationary points[a]



saddle point — maximum — minimum — quartic minimum

Stationary point if $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at (x_0, y_0) (necessary condition) (2.335)

Additional sufficient conditions

for maximum $\frac{\partial^2 f}{\partial x^2} < 0$, and $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$ (2.336)

for minimum $\frac{\partial^2 f}{\partial x^2} > 0$, and $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} > \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$ (2.337)

for saddle point $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} < \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$ (2.338)

[a]Of a function $f(x,y)$ at the point (x_0, y_0). Note that at, for example, a *quartic* minimum $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2} = 0$.

Differential equations

Laplace	$\nabla^2 f = 0$	(2.339)	f	$f(x,y,z)$
Diffusion[a]	$\frac{\partial f}{\partial t} = D\nabla^2 f$	(2.340)	D	diffusion coefficient
Helmholtz	$\nabla^2 f + \alpha^2 f = 0$	(2.341)	α	constant
Wave	$\nabla^2 f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2}$	(2.342)	c	wave speed
Legendre	$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + l(l+1)y = 0$	(2.343)	l	integer
Associated Legendre	$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right]y = 0$	(2.344)	m	integer
Bessel	$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + (x^2 - m^2)y = 0$	(2.345)		
Hermite	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2\alpha y = 0$	(2.346)		
Laguerre	$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1-x)\frac{\mathrm{d}y}{\mathrm{d}x} + \alpha y = 0$	(2.347)		
Associated Laguerre	$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1+k-x)\frac{\mathrm{d}y}{\mathrm{d}x} + \alpha y = 0$	(2.348)	k	integer
Chebyshev	$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - x\frac{\mathrm{d}y}{\mathrm{d}x} + n^2 y = 0$	(2.349)	n	integer
Euler (or Cauchy)	$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + ax\frac{\mathrm{d}y}{\mathrm{d}x} + by = f(x)$	(2.350)	a,b	constants
Bernoulli	$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)y^a$	(2.351)	p,q	functions of x
Airy	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = xy$	(2.352)		

[a]Also known as the "conduction equation." For thermal conduction, $f \equiv T$ and D, the thermal diffusivity, $\equiv \kappa \equiv \lambda/(\rho c_p)$, where T is the temperature distribution, λ the thermal conductivity, ρ the density, and c_p the specific heat capacity of the material.