2.5 Trigonometric and hyperbolic formulas

Trigonometric relationships

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B \tag{2.171}$$

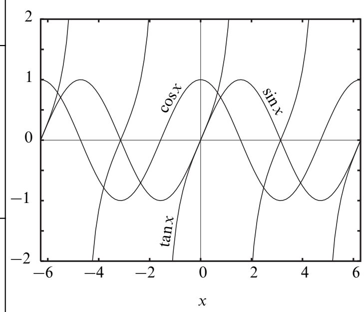
$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B \tag{2.172}$$

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B} \tag{2.173}$$

$$\cos A\cos B=\frac{1}{2}[\cos(A+B)+\cos(A-B)] \tag{2.174}$$

$$\sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)] \tag{2.175}$$

$$\sin A\sin B=\frac{1}{2}[\cos(A-B)-\cos(A+B)] \tag{2.176}$$



$$\cos^2 A+\sin^2 A=1 \tag{2.177}$$

$$\sec^2 A-\tan^2 A=1 \tag{2.178}$$

$$\csc^2 A-\cot^2 A=1 \tag{2.179}$$

$$\sin 2A=2\sin A\cos A \tag{2.180}$$

$$\cos 2A=\cos^2 A-\sin^2 A \tag{2.181}$$

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A} \tag{2.182}$$

$$\sin 3A=3\sin A-4\sin^3 A \tag{2.183}$$

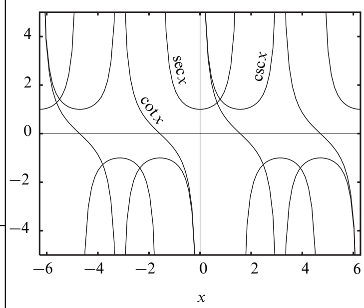
$$\cos 3A=4\cos^3 A-3\cos A \tag{2.184}$$

$$\sin A+\sin B=2\sin\frac{A+B}{2}\cos\frac{A-B}{2} \tag{2.185}$$

$$\sin A-\sin B=2\cos\frac{A+B}{2}\sin\frac{A-B}{2} \tag{2.186}$$

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \tag{2.187}$$

$$\cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2} \tag{2.188}$$



$$\cos^2 A=\frac{1}{2}(1+\cos 2A) \tag{2.189}$$

$$\sin^2 A=\frac{1}{2}(1-\cos 2A) \tag{2.190}$$

$$\cos^3 A=\frac{1}{4}(3\cos A+\cos 3A) \tag{2.191}$$

$$\sin^3 A=\frac{1}{4}(3\sin A-\sin 3A) \tag{2.192}$$

Hyperbolic relationships[a]

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y \tag{2.193}$$

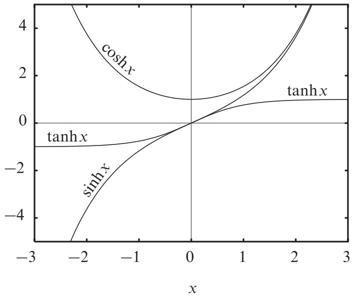
$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y \tag{2.194}$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y} \tag{2.195}$$

$$\cosh x \cosh y = \frac{1}{2}[\cosh(x+y) + \cosh(x-y)] \tag{2.196}$$

$$\sinh x \cosh y = \frac{1}{2}[\sinh(x+y) + \sinh(x-y)] \tag{2.197}$$

$$\sinh x \sinh y = \frac{1}{2}[\cosh(x+y) - \cosh(x-y)] \tag{2.198}$$



$$\cosh^2 x - \sinh^2 x = 1 \tag{2.199}$$

$$\operatorname{sech}^2 x + \tanh^2 x = 1 \tag{2.200}$$

$$\coth^2 x - \operatorname{csch}^2 x = 1 \tag{2.201}$$

$$\sinh 2x = 2 \sinh x \cosh x \tag{2.202}$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x \tag{2.203}$$

$$\tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x} \tag{2.204}$$

$$\sinh 3x = 3 \sinh x + 4 \sinh^3 x \tag{2.205}$$

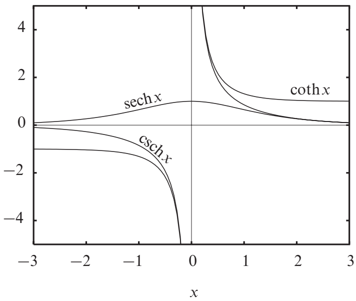
$$\cosh 3x = 4 \cosh^3 x - 3 \cosh x \tag{2.206}$$

$$\sinh x + \sinh y = 2 \sinh \frac{x+y}{2} \cosh \frac{x-y}{2} \tag{2.207}$$

$$\sinh x - \sinh y = 2 \cosh \frac{x+y}{2} \sinh \frac{x-y}{2} \tag{2.208}$$

$$\cosh x + \cosh y = 2 \cosh \frac{x+y}{2} \cosh \frac{x-y}{2} \tag{2.209}$$

$$\cosh x - \cosh y = 2 \sinh \frac{x+y}{2} \sinh \frac{x-y}{2} \tag{2.210}$$



$$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1) \tag{2.211}$$

$$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1) \tag{2.212}$$

$$\cosh^3 x = \frac{1}{4}(3 \cosh x + \cosh 3x) \tag{2.213}$$

$$\sinh^3 x = \frac{1}{4}(\sinh 3x - 3 \sinh x) \tag{2.214}$$

[a] These can be derived from trigonometric relationships by using the substitutions $\cos x \mapsto \cosh x$ and $\sin x \mapsto \mathbf{i} \sinh x$.

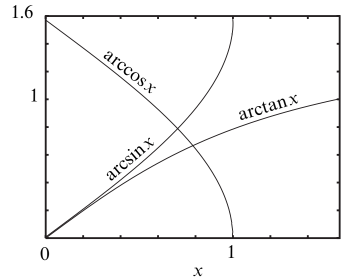
Trigonometric and hyperbolic definitions

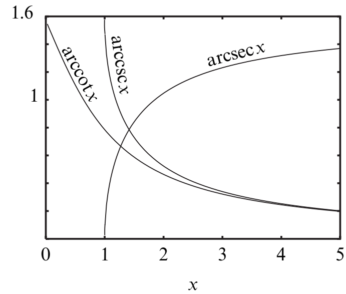
de Moivre's theorem	$(\cos x + \mathbf{i}\sin x)^n = \mathrm{e}^{\mathbf{i}nx} = \cos nx + \mathbf{i}\sin nx$	(2.215)	
$\cos x = \frac{1}{2}\left(\mathrm{e}^{\mathbf{i}x} + \mathrm{e}^{-\mathbf{i}x}\right)$ (2.216)		$\cosh x = \frac{1}{2}\left(\mathrm{e}^{x} + \mathrm{e}^{-x}\right)$	(2.217)
$\sin x = \frac{1}{2\mathbf{i}}\left(\mathrm{e}^{\mathbf{i}x} - \mathrm{e}^{-\mathbf{i}x}\right)$ (2.218)		$\sinh x = \frac{1}{2}\left(\mathrm{e}^{x} - \mathrm{e}^{-x}\right)$	(2.219)
$\tan x = \frac{\sin x}{\cos x}$ (2.220)		$\tanh x = \frac{\sinh x}{\cosh x}$	(2.221)
$\cos \mathbf{i}x = \cosh x$ (2.222)		$\cosh \mathbf{i}x = \cos x$	(2.223)
$\sin \mathbf{i}x = \mathbf{i}\sinh x$ (2.224)		$\sinh \mathbf{i}x = \mathbf{i}\sin x$	(2.225)
$\cot x = (\tan x)^{-1}$ (2.226)		$\coth x = (\tanh x)^{-1}$	(2.227)
$\sec x = (\cos x)^{-1}$ (2.228)		$\operatorname{sech} x = (\cosh x)^{-1}$	(2.229)
$\csc x = (\sin x)^{-1}$ (2.230)		$\operatorname{csch} x = (\sinh x)^{-1}$	(2.231)

Inverse trigonometric functions[a]

$\arcsin x = \arctan\left[\frac{x}{(1-x^2)^{1/2}}\right]$	(2.232)
$\arccos x = \arctan\left[\frac{(1-x^2)^{1/2}}{x}\right]$	(2.233)
$\operatorname{arccsc} x = \arctan\left[\frac{1}{(x^2-1)^{1/2}}\right]$	(2.234)
$\operatorname{arcsec} x = \arctan\left[(x^2-1)^{1/2}\right]$	(2.235)
$\operatorname{arccot} x = \arctan\left(\frac{1}{x}\right)$	(2.236)
$\arccos x = \frac{\pi}{2} - \arcsin x$	(2.237)

[a]Valid in the angle range $0 \le \theta \le \pi/2$. Note that $\arcsin x \equiv \sin^{-1} x$ etc.





Inverse hyperbolic functions

$\operatorname{arsinh}x \equiv \sinh^{-1}x = \ln\left[x+(x^2+1)^{1/2}\right]$	(2.238)	for all x
$\operatorname{arcosh}x \equiv \cosh^{-1}x = \ln\left[x+(x^2-1)^{1/2}\right]$	(2.239)	$x \geq 1$
$\operatorname{artanh}x \equiv \tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	(2.240)	$\lvert x\rvert < 1$
$\operatorname{arcoth}x \equiv \coth^{-1}x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$	(2.241)	$\lvert x\rvert > 1$
$\operatorname{arsech}x \equiv \operatorname{sech}^{-1}x = \ln\left[\frac{1}{x}+\frac{(1-x^2)^{1/2}}{x}\right]$	(2.242)	$0 < x \leq 1$
$\operatorname{arcsch}x \equiv \operatorname{csch}^{-1}x = \ln\left[\frac{1}{x}+\frac{(1+x^2)^{1/2}}{x}\right]$	(2.243)	$x \neq 0$