

2.3 Series, summations, and progressions

Progressions and summations

Arithmetic progression	$S_n = a + (a + d) + (a + 2d) + \cdots$		n number of terms
	$+ [a + (n - 1)d] \quad (2.102)$		S_n sum of n successive terms
	$= \frac{n}{2}[2a + (n - 1)d] \quad (2.103)$		a first term
	$= \frac{n}{2}(a + l) \quad (2.104)$		d common difference l last term
Geometric progression	$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \quad (2.105)$		r common ratio
	$= a \frac{1 - r^n}{1 - r} \quad (2.106)$		
	$S_\infty = \frac{a}{1 - r} \quad (r < 1) \quad (2.107)$		
Arithmetic mean	$\langle x \rangle_a = \frac{1}{n}(x_1 + x_2 + \cdots + x_n) \quad (2.108)$	$\langle \cdot \rangle_a$	arithmetic mean
Geometric mean	$\langle x \rangle_g = (x_1 x_2 x_3 \cdots x_n)^{1/n} \quad (2.109)$	$\langle \cdot \rangle_g$	geometric mean
Harmonic mean	$\langle x \rangle_h = n \left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} \right)^{-1} \quad (2.110)$	$\langle \cdot \rangle_h$	harmonic mean
Relative mean magnitudes	$\langle x \rangle_a \geq \langle x \rangle_g \geq \langle x \rangle_h \quad \text{if } x_i > 0 \text{ for all } i \quad (2.111)$		
Summation formulas	$\sum_{i=1}^n i = \frac{n}{2}(n + 1) \quad (2.112)$		i dummy integer
	$\sum_{i=1}^n i^2 = \frac{n}{6}(n + 1)(2n + 1) \quad (2.113)$		
	$\sum_{i=1}^n i^3 = \frac{n^2}{4}(n + 1)^2 \quad (2.114)$		
	$\sum_{i=1}^n i^4 = \frac{n}{30}(n + 1)(2n + 1)(3n^2 + 3n - 1) \quad (2.115)$		
	$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \ln 2 \quad (2.116)$		
	$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{2i - 1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4} \quad (2.117)$		
	$\sum_{i=1}^{\infty} \frac{1}{i^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6} \quad (2.118)$		
Euler's constant ^a	$\gamma = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n \right) \quad (2.119)$	γ	Euler's constant

^a $\gamma \simeq 0.577215664 \dots$

Power series

Binomial series ^a	$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$	(2.120)
Binomial coefficient ^b	${}^nC_r \equiv \binom{n}{r} \equiv \frac{n!}{r!(n-r)!}$	(2.121)
Binomial theorem	$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} b^k$	(2.122)
Taylor series (about a) ^c	$f(a+x) = f(a) + xf^{(1)}(a) + \frac{x^2}{2!}f^{(2)}(a) + \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(a) + \dots$	(2.123)
Taylor series (3-D)	$f(\mathbf{a}+\mathbf{x}) = f(\mathbf{a}) + (\mathbf{x} \cdot \nabla)f _{\mathbf{a}} + \frac{(\mathbf{x} \cdot \nabla)^2}{2!}f _{\mathbf{a}} + \frac{(\mathbf{x} \cdot \nabla)^3}{3!}f _{\mathbf{a}} + \dots$	(2.124)
Maclaurin series	$f(x) = f(0) + xf^{(1)}(0) + \frac{x^2}{2!}f^{(2)}(0) + \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \dots$	(2.125)

^aIf n is a positive integer the series terminates and is valid for all x . Otherwise the (infinite) series is convergent for $|x| < 1$.

^bThe coefficient of x^r in the binomial series.

^c $xf^{(n)}(a)$ is x times the n th derivative of the function $f(x)$ with respect to x evaluated at a , taken as well behaved around a . $(\mathbf{x} \cdot \nabla)^n f|_{\mathbf{a}}$ is its extension to three dimensions.

Limits

$n^c x^n \rightarrow 0$ as $n \rightarrow \infty$ if $ x < 1$ (for any fixed c)	(2.126)
$x^n/n! \rightarrow 0$ as $n \rightarrow \infty$ (for any fixed x)	(2.127)
$(1+x/n)^n \rightarrow e^x$ as $n \rightarrow \infty$	(2.128)
$x \ln x \rightarrow 0$ as $x \rightarrow 0$	(2.129)
$\frac{\sin x}{x} \rightarrow 1$ as $x \rightarrow 0$	(2.130)
If $f(a)=g(a)=0$ or ∞ then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f^{(1)}(a)}{g^{(1)}(a)}$ (l'Hôpital's rule)	(2.131)

Series expansions

$\exp(x)$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	(2.132)	(for all x)
$\ln(1+x)$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	(2.133)	$(-1 < x \leq 1)$
$\ln\left(\frac{1+x}{1-x}\right)$	$2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right)$	(2.134)	$(x < 1)$
$\cos(x)$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	(2.135)	(for all x)
$\sin(x)$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	(2.136)	(for all x)
$\tan(x)$	$x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} \cdots$	(2.137)	$(x < \pi/2)$
$\sec(x)$	$1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$	(2.138)	$(x < \pi/2)$
$\csc(x)$	$\frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \cdots$	(2.139)	$(x < \pi)$
$\cot(x)$	$\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \cdots$	(2.140)	$(x < \pi)$
$\arcsin(x)^a$	$x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^7}{7} \cdots$	(2.141)	$(x < 1)$
$\arctan(x)^b$	$\begin{cases} x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots & (x \leq 1) \\ \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots & (x > 1) \\ -\frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots & (x < -1) \end{cases}$	(2.142)	
$\cosh(x)$	$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$	(2.143)	(for all x)
$\sinh(x)$	$x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots$	(2.144)	(for all x)
$\tanh(x)$	$x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \cdots$	(2.145)	$(x < \pi/2)$

^a $\arccos(x) = \pi/2 - \arcsin(x)$. Note that $\arcsin(x) \equiv \sin^{-1}(x)$ etc.

^b $\operatorname{arccot}(x) = \pi/2 - \arctan(x)$.

Inequalities

Triangle inequality	$ a_1 - a_2 \leq a_1 + a_2 \leq a_1 + a_2 ;$	(2.146)
	$\left \sum_{i=1}^n a_i \right \leq \sum_{i=1}^n a_i $	(2.147)
Chebyshev inequality	if $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n$	(2.148)
	and $b_1 \geq b_2 \geq b_3 \geq \dots \geq b_n$	(2.149)
	then $n \sum_{i=1}^n a_i b_i \geq \left(\sum_{i=1}^n a_i \right) \left(\sum_{i=1}^n b_i \right)$	(2.150)
Cauchy inequality	$\left(\sum_{i=1}^n a_i b_i \right)^2 \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2$	(2.151)
Schwarz inequality	$\left[\int_a^b f(x)g(x) \, dx \right]^2 \leq \int_a^b [f(x)]^2 \, dx \int_a^b [g(x)]^2 \, dx$	(2.152)