Continuity and Differentiability
Part - 1

Assertion-Reasoning MCQs

Directions (Q. Nos. 64-78) Each of these questions contains two statements : Assertion (A) and Reason (R). Each of these questions also has four alternative choices, any one of which is the correct answer. You have to select one of the codes (a), (b), (c) and (d) given below.

(a) A is true, R is true; R is a correct explanation for A.
(b) A is true, R is true; R is not a correct explanation for A.
(c) A is true; R is False.
(d) A is false; R is true.

64. **Assertion (A)** The function $f(x) = \sqrt[3]{x}$ is continuous at all x except at $x = 0$.
Reason (R) The function $f(x) = [x]$ is continuous at $x = 2.99$, where [ ] is the greatest integer function.

65. **Assertion (A)** A function $f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$ is discontinuous at $x = 0$.
Reason (R) The function $f(x) = \begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$ is continuous for all values of x.

66. **Assertion (A)** $f(x)$ is continuous at $x = a$, iff $\lim_{x \to a} f(x)$ exists and equals to $f(a)$.
Reason (R) If $f(x)$ is continuous at a point, then $\frac{1}{f(x)}$ is also continuous at the point.

67. $f(x) = \begin{cases} \sin \pi x, & x < 1 \\ 0, & x = 1 \\ \dfrac{-\sin(x-1)}{x}, & x > 1 \end{cases}$

Assertion (A) $f(x)$ is discontinuous at $x = 1$.

Reason (R) $f(1) = 0$.

68. **Assertion (A)** The function $f(x) = \begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \le x \le 1 \\ 4x, & \text{if } x > 1 \end{cases}$ is continuous everywhere except at $x = 1$.

Reason (R) Polynomial and constant functions are always continuous.

69. $f(x) = \begin{cases} x + \pi, & \text{for } x \in [-\pi, 0) \\ \pi \cos x, & \text{for } x \in \left[0, \dfrac{\pi}{2}\right] \\ \left(x - \dfrac{\pi}{2}\right)^2, & \text{for } x \in \left(\dfrac{\pi}{2}, \pi\right] \end{cases}$

Consider the following statements

Assertion (A) The function $f(x)$ is continuous at $x = 0$.

Reason (R) The function $f(x)$ is continuous at $x = \pi/2$.

70. **Assertion (A)** The function $f(x) = |\cos x|$ is continuous function.

Reason (R) The function $f(x) = \cos|x|$ is a continuous function.

71. **Assertion (A)** The function defined by $f(x) = \cos(x^2)$ is a continuous function.

Reason (R) The sine function is continuous in its domain i.e. $x \in R$.

72. $f(x) = [x-1] + |x-2|$, where $[\cdot]$ denotes the greatest integer function.

Assertion (A) $f(x)$ is discontinuous at $x = 2$.

Reason (R) $f(x)$ is non derivable at $x = 2$.

73. **Assertion (A)** $f(x) = |x-3|$ is continuous at $x = 0$.

Reason (R) $f(x) = |x-3|$ is differentiable at $x = 0$.

74. **Assertion (A)** Every differentiable function is continuous but converse is not true.

Reason (R) Function $f(x) = |x|$ is continuous.

75. **Assertion (A)** If $f(x) = \dfrac{\sin(ax+b)}{\cos(cx+d)}$, then $f'(x) = a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\sec(cx+d)$

Reason (R) If $f(x) = \dfrac{u}{v}$, then $f'(x) = \dfrac{vu' - uv'}{v^2}$.

76. **Assertion (A)** $\dfrac{d}{dx} e^{\sin x} = e^{\sin x} (\cos x)$

Reason (R) $\dfrac{d}{dx} e^x = e^x$

77. **Assertion (A)** $\dfrac{d}{dx}\left(\sqrt{e^{\sqrt{x}}}\right) = \dfrac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}$.

Reason (R) $\dfrac{d}{dx}[\log(\log(x))] = \dfrac{1}{x \log x}$, $x > 1$

78. **Assertion (A)** If $f(x) = \log x$, then $f''(x) = -\dfrac{1}{x^2}$.

Reason (R) If $y = x^3 \log x$, then $\dfrac{d^2y}{dx^2} = x(5 + 6\log x)$.

ANSWER KEY

Assertion-Reasoning MCQs

64. (d) 65. (d) 66. (c) 67. (d) 68. (a) 69. (b) 70. (b) 71. (b) 72. (b) 73. (b)
74. (c) 75. (a) 76. (a) 77. (b) 78. (b)

SOLUTION

64. **Assertion** Given, $f(x) = \sqrt[3]{x}$ or $f(x) = (x)^{1/3}$

Now, we check the continuity of the function at $x = 0$.

$$\text{LHL} = f(0-0) = \lim_{h \to 0} f(0-h)$$
$$= \lim_{h \to 0} (0-h)^{1/3}$$
$$= (0-0)^{1/3} = 0$$

$$\text{RHL} = f(0+0) = \lim_{h \to 0} f(0+h)$$
$$= \lim_{h \to 0} (0+h)^{1/3} = (0+0)^{1/3} = 0$$

and $f(0) = (0)^{1/3} = 0$

$\because$ LHL = RHL = $f(0)$

So, function is continuous at $x = 0$.

Reason Given, $f(x) = [x]$, which is greatest integer function.

We know that, the greatest integer function is continuous for all x except integer values of x.
So, $f(x) = [x]$ is continuous at $x = 2.99$.

65. **Assertion** Here, $f(x)=\begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^2 \sin\frac{1}{x}.$$

Putting $x = 0 - h$ as $x \to 0^-, h \to 0$

$$\therefore \lim_{h \to 0} (0-h)^2 \sin\left(\frac{1}{0-h}\right) = \lim_{h \to 0}\left(-h^2 \sin\frac{1}{h}\right)$$

$[\because \sin(-\theta) = -\sin\theta]$

$$= -0 \times \sin(\infty)$$
$$= -0 \times (\text{a finite value between } -1 \text{ and } 1)$$
$$= 0$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 \sin\frac{1}{x}$$

Putting $x = 0 + h$, as $x \to 0^+, h \to 0$

$$\therefore \lim_{h \to 0} (0+h)^2 \sin\left(\frac{1}{0+h}\right) = \lim_{h \to 0} h^2 \sin\frac{1}{h}$$

$$= 0 \times \sin(\infty)$$
$$= 0 \times (\text{a finite value between } -1 \text{ and } 1)$$
$$= 0$$

Also, $f(0) = 0$

$\therefore$ LHL = RHL = $f(0)$.

Thus, $f(x)$ is continuous at $x = 0$.

Reason Here, $f(x)=\begin{cases} \sin x - \cos x, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (\sin x - \cos x)$$

Putting $x = 0 - h$ as $x \to 0^-$ when $h \to 0$

$$\therefore \lim_{h \to 0} [\sin(0-h) - \cos(0-h)]$$
$$= \lim_{h \to 0} (-\sin h - \cos h)$$
$$= 0 - 1 = -1$$

$$\text{RHL} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (\sin x - \cos x)$$

Putting $x = 0 + h$ as $x \to 0^+$ when $h \to 0$

$$\therefore \lim_{h \to 0} [\sin(0+h) - \cos(0+h)]$$
$$= \lim_{h \to 0} (\sin h - \cos h)$$
$$= 0 - 1 = -1$$

Also, $f(0) = -1$

$\therefore$ LHL = RHL = $f(0)$.

Thus, $f(x)$ is continuous at $x = 0$.

We know, when $x < 0$, $f(x) = \sin x - \cos x$ is continuous and when $x > 0$, $f(x) = \sin x - \cos x$ is also continuous.

Hence, $f(x)$ is continuous for all values of x.

66. **Assertion** We know that,

If $f(a) = \lim_{x \to a} f(x)$, then $f(x)$ is continuous at $x = a$, while both hand must exist.

Reason If $f(x)$ is continuous at a point, then it is not necessary that $\frac{1}{f(x)}$ is also continuous at that point.

e.g. $f(x) = x$ is continuous at $x = 0$ but $f(x) = \frac{1}{x}$ is not continuous at $x = 0$.

67. **Assertion** $f(x)=\begin{cases} \sin \pi x, & x < 1 \\ 0, & x = 1 \\ -\frac{\sin(x-1)}{x}, & x > 1 \end{cases}$

Also, $\text{LHL} = \lim_{x \to 1^-} f(x)$

$$= \lim_{h \to 0} f(1-h) = \lim_{h \to 0} \sin(\pi - \pi h)$$
$$= \lim_{h \to 0} \sin(\pi h) = \sin 0 = 0$$

$$\text{RHL} = \lim_{x \to 1^+} f(x)$$
$$= \lim_{h \to 0} f(1+h)$$
$$= \lim_{h \to 0} \frac{-\sin(1+h-1)}{(1+h)}$$
$$= -\lim_{h \to 0} \frac{\sin h}{1+h} = 0$$

and $f(1) = 0$

$\therefore$ LHL = RHL = $f(1)$

$\Rightarrow f(x)$ is continuous at $x = 1$.

$\therefore$ Assertion is false.

Reason It is clear that $f(1) = 0$

$\therefore$ Reason is true.

68. **Assertion** Here, $f(x)=\begin{cases} 2x, & \text{if } x < 0 \\ 0, & \text{if } 0 \leq x \leq 1 \\ 4x, & \text{if } x > 1 \end{cases}$

For $x < 0$, $f(x) = 2x$; $0 < x < 1$, $f(x) = 0$ and $x > 1$, $f(x) = 4x$ are polynomial and constant functions, so it is continuous in the given interval.

So, we have to check the continuity at $x = 0$ and 1.

At $x = 0$,

$$\text{LHL} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x)$$

Putting $x = 0 - h$ as $x \to 0^-, h \to 0$

$\therefore \lim_{h\to 0} [2(0-h)] = \lim_{h\to 0} (-2h) = -2 \times 0 = 0,$

$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (0) = 0$

Also, $f(0) = 0$

$\therefore \quad \text{LHL} = \text{RHL} = f(0)$

Thus, $f(x)$ is continuous at $x = 0$.

At $x = 1$,

$\text{LHL} = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (0) = 0,$

$\text{RHL} = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4x)$

Putting $x = 1 + h$ as $x \to 1^+$ when $h \to 0$

$\text{RHL} = \lim_{h\to 0} 4(1+h) = \lim_{h\to 0} (4 + 4h)$

$= 4 + 4 \times 0 = 4$

$\therefore \text{LHL} \neq \text{RHL}.$

Thus, $f(x)$ is continuous everywhere except at $x = 1$.

69. **Assertion** $\text{LHL} = \lim_{x\to 0^-} f(x)$

$= \lim_{x\to 0}(x + \pi) = \pi$

$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{x\to 0} \pi \cos x$

$= \pi \cos(0) = \pi$

Also, $f(0) = \pi \cos(0) = \pi$

Hence, $f(x)$ is continuous at $x = 0$.

$\therefore$ Assertion is true.

Reason Now, for $x = \frac{\pi}{2}$

$\text{LHL} = \lim_{x\to \pi/2^-} f(x) = \lim_{x\to \pi/2} \pi \cos x$

$= \pi \cos \frac{\pi}{2} = 0$

$\text{RHL} = \lim_{x\to \pi/2^+} f(x) = \lim_{x\to \pi/2} \left(x - \frac{\pi}{2}\right)^2$

$= \left(\frac{\pi}{2} - \frac{\pi}{2}\right)^2 = 0$

Also, $f\left(\frac{\pi}{2}\right) = \pi \cos \frac{\pi}{2} = 0$

Hence, $f(x)$ is continuous at $x = \frac{\pi}{2}$.

$\therefore$ Reason is true.

70. **Assertion** We have, $f(x) = |\cos x|$

$$= \begin{cases} \cos x, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Continuity at $x = 0$,

$\text{LHL} = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \cos(0-h) = \cos 0 = 1$

$\text{RHL} = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \cos(0+h)$

$= \lim_{h\to 0} \cos h = \cos 0 = 1$

and $f(0) = 1$

$\therefore \text{LHL} = \text{RHL} = f(0)$

So, $f(x)$ is continuous at $x = 0$.

Hence, $f(x)$ is continuous everywhere.

Reason We have, $f(x) = \cos|x|$

$$= \begin{cases} \cos x, & x \geq 0 \\ \cos(-x), & x < 0 \end{cases}$$

$$= \begin{cases} \cos x, & x \geq 0 \\ \cos x, & x < 0 \end{cases}$$

$= \cos x, \ x \in R$

But $\cos x$ is always continuous in their domain.

Hence, $f(x)$ is continuous everywhere.

Hence, both Assertion and Reason are true, but Reason is not the correct explanation of Assertion.

71. **Assertion** We have, $f(x) = \cos(x^2)$

At $x = c$,

$\text{LHL} = \lim_{h\to 0} \cos(c-h)^2 = \cos c^2$

$\text{RHL} = \lim_{h\to 0} \cos(c+h)^2 = \cos c^2$

and $f(c) = \cos c^2$

$\therefore \text{LHL} = \text{RHL} = f(c)$

So, $f(x)$ is continuous at $x = c$.

Hence, $f(x)$ is continuous for every value of x.

Hence, both Assertion and Reason are true and Reason is not the correct explanation of Assertion.

72. **Assertion**

$\text{LHL} = \lim_{x\to 2^-} f(x) = \lim_{h\to 0} f(2-h)$

$= \lim_{h\to 0} [2-h-1] + |2-h-2|$

$= \lim_{h\to 0} [1-h] + |-h| = \lim_{h\to 0} (0+h) = 0$

and $\quad f(2) = [2-1] + |2-2| = [1] + 0 = 1$

$\therefore \quad \text{LHL} \neq f(2)$

$\Rightarrow f(x)$ is discontinuous at $x = 2$.

Reason

$$Lf'(2) = \lim_{h\to 0}\frac{f(2-h)-f(2)}{-h}$$

$$= \lim_{h\to 0}\frac{[2-h-1]+|2-h-2|-[2-1]-|2-2|}{-h}$$

$$= \lim_{h\to 0}\frac{0+h-1-0}{-h} \quad [\because \lim_{h\to 0}[1-h]=0]$$

$$= \lim_{h\to 0}\left(1-\frac{1}{h}\right) = -\infty \quad \text{(not defined)}$$

$\therefore f(x)$ is not differentiable at $x = 2$.

Hence, both Assertion and Reason are true and Reason is not a correct explanation of Assertion.

73. **Assertion** $\because f(x) = |x-3| = \begin{cases} x-3, & x \geq 3 \\ 3-x, & x < 3 \end{cases}$

$$\therefore \quad \text{LHL} = \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0}(3+h) = 3$$

$$\text{RHL} = \lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0}(3-h) = 3$$

and $f(0) = 3 - 0 = 3$

$\Rightarrow$ LHL = RHL = $f(0)$

So, $f(x)$ is continuous at $x = 0$.

Reason Now, LHD = $f'(0^-)$

$$= \lim_{h\to 0}\frac{f(0)-f(0-h)}{h}$$

$$= \lim_{h\to 0}\frac{3-(3-h)}{h} = 1$$

and RHD $= f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to 0}\frac{3+h-3}{h} = 1$$

$\Rightarrow$ LHD = RHD

$\therefore f(x)$ is differentiable at $x = 0$.

Hence, both Assertion and Reason are true.

74. **Assertion** It is a true statement.

Reason We have, $f(x) = |x|$

At $x = 0$,

$$\text{LHL} = \lim_{h\to 0^-}\frac{f(0-h)-f(0)}{-h}$$

$$= \lim_{h\to 0^-}\frac{|0-h|-0}{-h}$$

$$= \lim_{h\to 0^-}\frac{h}{-h} = -1$$

and RHL $= \lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to 0^+}\frac{|0+h|-0}{h} = \lim_{h\to 0^+}\frac{h}{h} = 1$$

Here, LHD $\neq$ RHD, hence $f(x)$ is not continuous at $x = 0$.

75. Let $y = \dfrac{\sin(ax+b)}{\cos(cx+d)}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin(ax+b)}{\cos(cx+d)}\right)$$

$$= \frac{\cos(cx+d)\frac{d}{dx}\{\sin(ax+b)\} - \sin(ax+b)\frac{d}{dx}\cos(cx+d)}{[\cos(cx+d)]^2}$$

[by quotient rule]

$$= \frac{\cos(cx+d)\cos(ax+b)(a+0) + \sin(ax+b)\sin(cx+d)(c+0)}{\cos^2(cx+d)}$$

$$\left[\begin{aligned} &\text{by chain rule,} \\ &\frac{d}{dx}\sin(ax+b) = \cos(ax+b)\frac{d}{dx}(ax+b) \\ &\qquad = \cos(ax+b)\times(a\times 1+0) \\ &\frac{d}{dx}\cos(cx+d) = -\sin(cx+d)\frac{d}{dx}(cx+d) \\ &\qquad = -\sin(cx+d)\times(c\times 1+0) \end{aligned}\right]$$

$$= \frac{a\cos(cx+d)\cos(ax+b) + c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{a\cos(cx+d)\cos(ax+b)}{\cos^2(cx+d)} + \frac{c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$$

$$= \frac{a\cos(ax+b)}{\cos(cx+d)} + \frac{c\sin(ax+b)\sin(cx+d)}{\cos(cx+d)\cos(cx+d)}$$

$$= a\cos(ax+b)\sec(cx+d) + c\sin(ax+b)\tan(cx+d)\sec(cx+d)$$

76. **Assertion** Let $y = e^{\sin x}$.

Using chain rule, we have

$$\frac{dy}{dx} = e^{\sin x}\cdot(\cos x)$$

$$= \cos x\, e^{\sin x}$$

Reason $\frac{d}{dx}(e^x) = e^x . \frac{d(x)}{dx} = e^x \times 1 = e^x$

Hence, both Assertion and Reason are true, but Reason is the correct explanation of Assertion.

77. **Assertion** Let $y = (e^{\sqrt{x}})^{1/2}$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{2}(e^{\sqrt{x}})^{\frac{1}{2}-1} \frac{d}{dx} e^{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}(e^{\sqrt{x}})^{-\frac{1}{2}} . e^{\sqrt{x}} . \frac{d}{dx}(\sqrt{x})$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \frac{e^{\sqrt{x}}}{\sqrt{e^{\sqrt{x}}}} \times \frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{4\sqrt{x}\sqrt{e^{\sqrt{x}}}} = \frac{e^{\sqrt{x}}}{4\sqrt{xe^{\sqrt{x}}}}$$

Reason Let $y = \log(\log x)$

On differentiating both sides w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(\log(\log x)) = \frac{1}{\log x}\left\{\frac{d}{dx}(\log x)\right\}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\log x} . \frac{1}{x} = \frac{1}{x \log x}, \; x > 1$$

Hence, both Assertion and Reason are true, but Reason is not the correct explanation of Assertion.

78. **Assertion** Let $y = \log x$

On differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{x}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$$

Reason Let $y = x^3 \log x$

On differentiating twice w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(x^3 \log x)$$

$$= x^3 \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x^3)$$

$$= x^3\left(\frac{1}{x}\right) + (\log x)(3x^2)$$

$$= x^2(1 + 3\log x)$$

[using product rule]

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\{x^2(1 + 3\log x)\}$$

$$= x^2\left(0 + \frac{3}{x}\right) + (1 + 3\log x)(2x)$$

$$= 3x + 2x(1 + 3\log x)$$

$$= x(5 + 6\log x)$$

Hence, both Assertion and Reason are true, but Reason is not the correct explanation of Assertion.